NP-Completeness

CPSC 6109 - Algorithms Analysis and Design

Dr. Hyrum D. Carroll

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Complexity Classes

Classes:

- **P**: Problems that are solvable in polynomial time: $O(n^k)$
- **NP**: Verifiable in polynomial time
  (verifiable means we can check the answer)

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(because we can more than check an answer, we can solve it in polynomial time)
The open question is if it's $P \subset NP$

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- **NP**: Verifiable in polynomial time
  (verifiable means we can check the answer)
- **NP-Complete**: as hard as any other problem in NP

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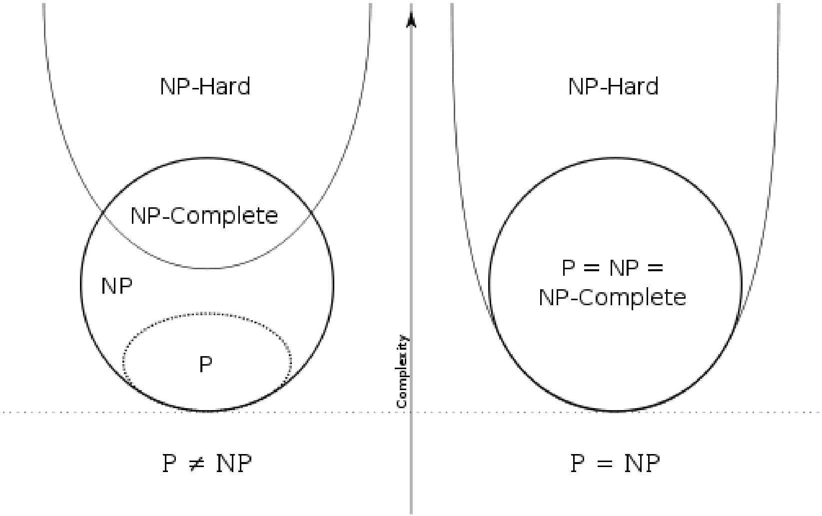
Complexity Classes (illustrated)



Source: Wikimedia Commons, user: Behnam Esfahbod

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- Is it in P?
- Can we develop an algorithm that runs in polynomial time?

Exercise

- Is determining the solution to a linear programming problem in P or NP?

Exercise: Solution

- Is determining the solution to a linear programming problem in P, NP or NP-Complete?
- Cast the question as a yes-no question:
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- So, in P
- But integer linear programming is NP-Complete :)

How knowing about complexity can help you

- If you're asked to implement a solution to a problem that is NP-Complete, don't waste your time coming up with an exact solution, but focus on:
  - Approximations (Chapter 35)
  - Heuristics
  - Accepting that an exponential run-time is the best you can do
  - Determine if you can solve just a subset of the problems efficiently

Showing a Problem is NP-Complete

- Instead of how easy a problem is, we're saying, **how hard the problem is**
- Instead of proving an efficient algorithm, we showing that, **no efficient algorithm is likely to exist**

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- Example:
  - `Shortest Paths` $\rightarrow$ `Path`
  - Given a graph and weights **and threshold** k, is there a path between vertices u and v **that has at most** k **edges**

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- Often, solving the optimization problem will solve the decision problem (because it's a subset of it)
- So, the decision problem version is easier
- If we can prove that the decision problem is hard, then we can prove that the optimization problem is hard

Reductions

- Almost every NP-Complete proof makes a reduction of one problem to another

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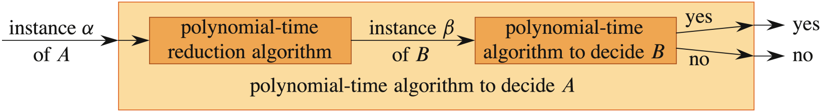

Figure 34.1 from Introduction to Algorithms 4^{th} Edition

Polynomial-Time Reductions (Pseudocode)

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Boolean B( β ); // known solution

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- We use the easiness of B to prove the easiness of A

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Exercise: One NP-Complete Problem

- Choose one of Karp's 21 NP-Complete problems
- Describe your algorithm
- Describe how we know it's NP-Complete

Exercise: Heirarchy of Karp's 21 NP-Complete Problems

- Draw a heirarchy (showing which problem was reduce to a known NP-complete problem) for each one of Karp's 21 NP-complete problems