Linear Programming

CPSC 6109 - Algorithms Analysis and Design

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The most successful men in the end are those whose success is the result of steady accretion.

– Alexander Graham Bell

Overview

Linear programming is finding a set of coefficients that the maximize (or minimize) a linear function subject to constraints

Example:

Maximize:

$$3x_1 + 5x_2 \tag{1}$$

Subject to:

$$x_1 + x_2 \leq 4 \tag{2}$$

$$x_1 + 3x_2 \leq 6 \tag{3}$$

$$x_1 \geq 0 \tag{4}$$

$$x_2 \geq 0 \tag{5}$$

Applications

- Airline crew scheduling
- Transportation network planning
- Communication network planning
- Oil exploration and refining
- Industrial production optimization

Terms

- Linear function:

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n = \sum_{j=1}^{n} a_j x_j \qquad (6)$$

- Linear constraints:
  - Linear equality

$$f(x_1, x_2, \ldots, x_n) = b \qquad (7)$$

  - Linear inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b \qquad (8)$$

$$f(x_1, x_2, \ldots, x_n) \geq b \qquad (9)$$

Terms (cont'd)

- Feasible region
  - convex region
  - *i.e.*, blue area
- Objective value/feasible solution
  - any point in the feasible region

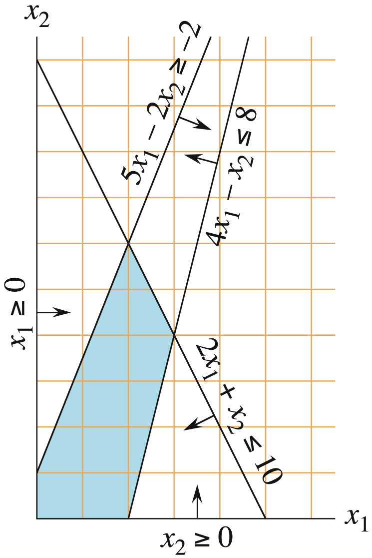

Figure 29.2a from Introduction to Algorithms 4^{th} Edition

Terms (cont'd 2)

- Objective function
  - linear function we're maximizing/minimizing
- Optimal solution

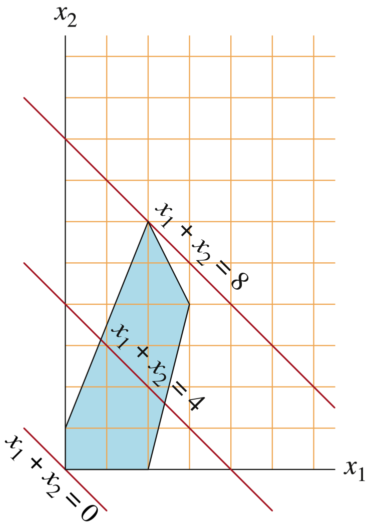

Figure 29.2b from Introduction to Algorithms 4^{th} Edition

Simplex Algorithm

Background

Discovered by US mathematician George Dantzig in 1940
Worst-case run-time is known to be exponential, that rarely happens in real-world applications

Standard Form

All constraints are inequalities

Maximize (objective function):

$$3x_1 + x_2 + 2x_3 \tag{10}$$

Subject to (constraints):

$$x_1 + x_2 + 3x_3 \leq 30 \tag{11}$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{12}$$

$$4x_1 + x_2 + 2x_3 \leq 36 \tag{13}$$

Nonnegativity constraints:

$$x_1, x_2, x_3 \geq 0 \tag{14}$$

Convert to Standard Form

Four conversions to convert to standard form (See Section 29.1):

1. Convert minimization objective function to maximization objective function
2. Replace nonnegativity constraints
3. Equality constraints to inequality constraints
4. $\geq$ constraints to $\leq$ constraints

Convert to Standard Form (Step 1)

Four conversions to convert to standard form (See Section 29.1):

1. **Convert minimization objective function to maximization objective function**
2. Replace nonnegativity constraints
3. Equality constraints to inequality constraints
4. $\geq$ constraints to $\leq$ constraints

Convert to Standard Form: Minimization to Maximization

Negate the coefficients in the objective function
Example:

Minimize:

$$10x_1 - 5x_2$$

Convert to Standard Form: Minimization to Maximization

Negate the coefficients in the objective function
Example:

Maximize:

$$-10x_1 + 5x_2$$

Convert to Standard Form (Step 2)

Four conversions to convert to standard form (See Section 29.1):

1. Convert minimization objective function to maximization objective function
2. **Replace nonnegativity constraints**
3. Equality constraints to inequality constraints
4. $\geq$ constraints to $\leq$ constraints

Convert to Standard Form: Nonnegativity Constraints

Replace each variable, x_i, that does not have a nonnegativity constraint with $x_i' - x_i''$
Example:
Maximize:

$$-10x_1 + 5x_2$$

Subject to:

$$x_1 + x_2 = 20$$

$$x_1 \leq 12$$

$$x_2 \leq 16$$

$$x_1 \geq 0$$

x_2 does not have a nonnegativity constraint, replace with $x_2' - x_2''$

Convert to Standard Form: Nonnegativity Constraints (2)

Replace each variable, x_i, that does not have a nonnegativity constraint with $x_i' - x_i''$
Example:
Maximize:

$$-10x_1 + 5x_2' - 5x_2''$$

Subject to:

$$x_1 + x_2' - x_2'' = 20$$

$$x_1 \leq 12$$

$$x_2' - x_2'' \leq 16$$

$$x_1, x_2', x_2'' \geq 0$$

Convert to Standard Form (Step 3)

Four conversions to convert to standard form (See Section 29.1):

1. Convert minimization objective function to maximization objective function
2. Replace nonnegativity constraints
3. **Equality constraints to inequality constraints**
4. $\geq$ constraints to $\leq$ constraints

Convert to Standard Form: Equality to Inequality

For each constraint:

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n = b$$

replace with:

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n \leq b$$

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n \geq b$$

Example:
Maximize:

$$-10x_1 + 5x_2' - 5x_2''$$

Subject to:

$$x_1 + x_2' - x_2'' = 20$$

$$x_1 \leq 12$$

$$x_2' - x_2'' \leq 16$$

$$x_1, x_2', x_2'' \geq 0$$

Convert to Standard Form: Equality to Inequality (2)

For each constraint:

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots a_nx_n = b$$

replace with:

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots a_nx_n \leq b$$

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots a_nx_n \geq b$$

Example:
Maximize:

$$-10x_1 + 5x_2' - 5x_2''$$

Subject to:

$$x_1 + x_2' - x_2'' \leq 20$$

$$x_1 + x_2' - x_2'' \geq 20$$

$$x_1 \leq 12$$

$$x_2' - x_2'' \leq 16$$

$$x_1, x_2', x_2'' \geq 0$$

Convert to Standard Form (Step 4)

Four conversions to convert to standard form (See Section 29.1):

1. Convert minimization objective function to maximization objective function
2. Replace nonnegativity constraints
3. Equality constraints to inequality constraints
4. $\geq$ **constraints to** $\leq$ **constraints**

Convert to Standard Form: $\geq$ to $\leq$

For each constraint:

$$\sum_{j=1}^{n} a_j x_j \geq b \tag{15}$$

replace with:

$$\sum_{j=1}^{n} -a_j x_j \leq -b \tag{16}$$

Example:
Maximize:

$$-10x_1 + 5x_2' - 5x_2''$$

Subject to:

$$x_1 + x_2' - x_2'' \leq 20$$
$$x_1 + x_2' - x_2'' \geq 20$$
$$x_1 \leq 12$$
$$x_2' - x_2'' \leq 16$$
$$x_1, x_2', x_2'' \geq 0$$

Convert to Standard Form: $\geq$ to $\leq$ (2)

For each constraint:

$$\sum_{j=1}^{n} a_j x_j \geq b \tag{15}$$

replace with:

$$\sum_{j=1}^{n} -a_j x_j \leq -b \tag{16}$$

Example:
Maximize:

$$-10x_1 + 5x_2' - 5x_2''$$

Subject to:

$$x_1 + x_2' - x_2'' \leq 20$$
$$-x_1 - x_2' + x_2'' \leq -20$$
$$x_1 \leq 12$$
$$x_2' - x_2'' \leq 16$$
$$x_1, x_2', x_2'' \geq 0$$

Standard Form → Slack Form

Simplex algorithm works with equalities
Given:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots a_{in}x_n \leq b_i \tag{17}$$

can be converted into:

$$s_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \ldots a_{in}x_n \tag{18}$$

$$s_i \geq 0 \tag{19}$$

where s_i is a *slack variable* (capturing the difference between the two sides in the inequality)

Instead of s_i we'll use x_{n+i}

Slack Form

All constraints are equalities (except when variables are required to be nonnegative)

Maximize:

$$z = 3x_1 + x_2 + 2x_3 \tag{20}$$

Subject to:

$$x_4 = 30 - x_1 - x_2 - 3x_3 \tag{21}$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \tag{22}$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3 \tag{23}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \tag{24}$$

Slack Form: Terms

Maximize:

$$z = 3x_1 + x_2 + 2x_3$$

Subject to:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic variables are on the left side and **non-basic variables** are on the right side of the equations
(the sets of basic and non-basic variables will change)

Standard Form → Slack Form Exercise

Convert the following standard form linear program into slack form:

Maximize:

$$2x_1 - 6x_3$$

Subject to:

$$x_1 + x_2 - x_3 \leq 7$$

$$-3x_1 + x_2 \leq -8$$

$$x_1 - 2x_2 - 2x_3 \leq 0$$

$$x_1, x_2, x_3 \geq 0$$

Standard Form $\rightarrow$ Slack Form Exercise

Solution:

$$
\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Slack Form

Maximize:

$$z = 3x_1 + x_2 + 2x_3$$

Subject to:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

3 equations and 6 unknowns (so infinite number of possibilities)

Slack Form

Maximize:

$$z = 3x_1 + x_2 + 2x_3$$

Subject to:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

3 equations and 6 unknowns (so infinite number of possibilities)
Focus on basic solution: set all variables on the right-hand side set to 0.
Here: $x_1 = 0, x_2 = 0, x_3 = 0$, so $x_4 = 30, x_5 = 24, x_6 = 36$ and $z = 0$

Goals

The simplex algorithm re-writes the set of equations and the object function so that there's different variables in the objective function.
Re-writing the equations changes the basic solution (and therefore the objective function).
Re-writing the equations does not change the system or underlying problem.
Each iteration, increase the objective function by re-writing the equations.

Pivoting

Pivoting:

1. Select a non-basic variable (x_e, e for entering) whose coefficient in the objective function is positive
2. Increase x_e as much as possible
3. Switch x_e with a basic variable, x_l (l for leaving)

Determining x_e

$$z = 3x_1 + x_2 + 2x_3$$

x_1 has the largest positive coefficient in the objective function

$x_e = x_1$

Determining x_l

To maximize the objective function using x_1:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

$x_1 \leq 30$

Determining x_l

To maximize the objective function using x_1:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

$x_1 \leq 30$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$x_1 \leq \frac{24}{2} = 12$

Determining x_l

To maximize the objective function using x_1:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

$x_1 \leq 30$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$x_1 \leq \frac{24}{2} = 12$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$x_1 \leq \frac{36}{4} = 9$

Determining x_l

To maximize the objective function using x_1:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

$x_1 \leq 30$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$x_1 \leq \frac{24}{2} = 12$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$x_1 \leq \frac{36}{4} = 9$

Choose the tightest constraint, so $x_l = x_6$.

First Pivot

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Solve for x_1:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \tag{25}$$

Now, re-write the other equations, substituting for x_1 using Eqn. 25:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \tag{26}$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \tag{27}$$

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \tag{28}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Verify nothing changed

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

In the beginning, with
$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36$, $z = 0$. Is that still true?

Verify nothing changed

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

In the beginning, with
$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36$, $z = 0$. Is that still true? Yes!
The thing that changed was the basic solution (set non-basic variables to 0): $x_2 = 0, x_3 = 0, x_6 = 0$ so
$z = 27, x_1 = 9, x_4 = 21, x_5 = 6$.

Second Iteration

Continue to increase objective function.
Determine x_e:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$

x_3 has the largest positive coefficient in the objective function
$x_e = x_3$

Determining x_l

To maximize the objective function using x_3:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$x_3 \leq \frac{9}{\frac{1}{2}} = 18$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

$x_3 \leq \frac{21}{\frac{5}{2}} = \frac{42}{5} = 8.4$

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

$x_3 \leq \frac{6}{4} = \frac{3}{2} = 1.5$

Choose the tightest constraint, so $x_l = x_5$.

Second Pivot

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

Solve for x_3:

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{x_5}{4} + \frac{x_6}{8} \quad (29)$$

Now, re-write the other equations, substituting for x_3 using Eqn. 29:

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \quad (30)$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \quad (31)$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \quad (32)$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \quad (33)$$

End of Second Iteration

New basic solution (set non-basic variables to 0):
$x_2 = 0, x_5 = 0, x_6 = 0$ so
$z = \frac{111}{4} = 27.75, x_1 = \frac{33}{4}, x_3 = \frac{3}{2}, x_4 = \frac{69}{4}$.

Third Iteration

Continue to increase objective function.
Determine x_e:

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \tag{34}$$

x_2 is the only way to increase the objective function
$x_e = x_2$

Determining x_l

To maximize the objective function using x_2:

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$x_2 \leq \frac{\frac{33}{4}}{\frac{1}{16}} = 132$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$x_2 \leq \frac{\frac{3}{2}}{\frac{3}{8}} = 4$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

$x_2 \leq \infty$

Choose the tightest constraint, so $x_l = x_3$.

Third Pivot

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

Solve for x_2:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \tag{35}$$

Now, re-write the other equations, substituting for x_2 using Eqn. 35:

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \tag{36}$$

$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \tag{37}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \tag{38}$$

$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \tag{39}$$

End of Third Iteration

New basic solution (set non-basic variables to 0):
$x_3 = 0, x_5 = 0, x_6 = 0$, so $z = 28, x_1 = 8, x_2 = 4, x_4 = 18$.

Exercise

Apply the simplex algorithm to solve the following program:

Maximize:

$$18x_1 + 12.5x_2$$

Subject to:

$$x_1 + x_2 \leq 20$$

$$x_1 \leq 12$$

$$x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Exercise: Solution

$$z = 18x_1 + 12.5x_2$$

$$x_3 = 20 - x_1 - x_2$$

$$x_4 = 12 - x_1$$

$$x_5 = 16 - x_2$$

$$x_{1\ldots6} \geq 0$$

Basic solution: $x_1 = 0, x_2 = 0, x_3 = 20, x_4 = 12, x_5 = 16$, so $z = 0$

Exercise: Solution

$$z = 18x_1 + 12.5x_2$$

$$x_e = x_1$$

$$x_3 = 20 - x_1 - x_2$$

$x_1 \leq 20$

$$x_4 = 12 - x_1$$

$x_1 \leq 12$

$$x_5 = 16 - x_2$$

$x_1 = ?$

$$x_l = x_4$$

Exercise: Solution

$$x_1 = 12 - x_4$$

$$z = 216 + 12.5x_2 - 18x_4$$

$$x_1 = 12 - x_4$$

$$x_3 = 8 - x_2 + x_4$$

$$x_5 = 16 - x_2$$

Original basic solution (still holds):
$x_1 = 0, x_2 = 0, x_3 = 20, x_4 = 12, x_5 = 16$, so $z = 0$

Exercise: Solution

$$z = 216 + 12.5x_2 - 18x_4$$

$$x_e = x_2$$

$$x_1 = 12 - x_4$$

$x_2 = ?$

$$x_3 = 8 - x_2 + x_4$$

$x_2 \leq 8$ (most restrictive)

$$x_5 = 16 - x_2$$

$x_2 \leq 16$

$$x_l = x_3$$

$$x_2 = 8 - x_3 + x_4$$

Exercise: Solution

$$x_2 = 8 - x_3 + x_4$$

$$z = 316 - 12.5x_3 - 5.5x_4$$

$$x_1 = 12 - x_4$$

$$x_2 = 8 - x_3 + x_4$$

$$x_5 = 8 + x_3 - x_4$$

Basic solution:
$x_1 = 12, x_2 = 8, x_3 = 0, x_4 = 0, x_5 = 8,$ so $z = 316$