Linear Programming
CPSC 6109 - Advanced Algorithms

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The most successful men in the end are those whose success is the result of steady accretion.

– Alexander Graham Bell
Overview

Linear programming is finding a set of coefficients that the maximize (or minimize) a linear function subject to constraints

Example:
Maximize:
\[ 3x_1 + 5x_2 \] (1)

Subject to:
\[ x_1 + x_2 \leq 4 \] (2)
\[ x_1 + 3x_2 \leq 6 \] (3)
\[ x_1 \geq 0 \] (4)
\[ x_2 \geq 0 \] (5)
Applications

- Airline crew scheduling
- Transportation network planning
- Communication network planning
- Oil exploration and refining
- Industrial production optimization
Terms

- Linear function:

\[ f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n = \sum_{j=1}^{n} a_jx_j \]  

(6)

- Linear constraints:
  - Linear equality
    \[ f(x_1, x_2, \ldots, x_n) = b \]  
    (7)
  - Linear inequalities
    \[ f(x_1, x_2, \ldots, x_n) \leq b \]  
    (8)
    \[ f(x_1, x_2, \ldots, x_n) \geq b \]  
    (9)
Terms (cont’d)

- Feasible region
  - convex region
  - e.g., gray area
- Objective value/feasible solution
  - any point in the feasible region

Figure 29.2a from Introduction to Algorithms 3rd Edition
Terms (cont’d 2)

- Objective function
  - linear function we’re maximizing/minimizing
- Optimal solution

Figure 29.2b from Introduction to Algorithms 3rd Edition
Background

Discovered by US mathematician George Dantzig in 1940.
Worst-case run-time is known to be exponential, that rarely happens in real-world applications.
Standard Form

All constraints are inequalities Maximize (objective function):

\[ 3x_1 + x_2 + 2x_3 \quad (10) \]

Subject to (constraints):

\[ x_1 + x_2 + 3x_3 \leq 30 \quad (11) \]
\[ 2x_1 + 2x_2 + 5x_3 \leq 24 \quad (12) \]
\[ 4x_1 + x_2 + 2x_3 \leq 36 \quad (13) \]

Positivity constraints:

\[ x_1, x_2, x_3 \geq 0 \quad (14) \]
Convert to Standard Form

Verify conditions in Section 29.1 are met to convert to standard form:
Standard Form → Slack Form

Simplex algorithm works with equalities

Given:

\[ a_{i1}x_1 + a_{i2}x_2 + \ldots a_{in}x_n \leq b_i \]  \hspace{1cm} (15)

can be converted into:

\[ s_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \ldots - a_{in}x_n \]  \hspace{1cm} (16)

\[ s_i \geq 0 \]  \hspace{1cm} (17)

where \( s_i \) is a *slack variable* (capturing the difference between the two sides in the inequality)

Instead of \( s_i \) we’ll use \( x_{n+i} \)
Slack Form

All constraints are equalities (except when variables are required to be positive)

Maximize:

\[ z = 3x_1 + x_2 + 2x_3 \]  \hspace{1cm} (18)

Subject to:

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]  \hspace{1cm} (19)
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]  \hspace{1cm} (20)
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]  \hspace{1cm} (21)

\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]  \hspace{1cm} (22)
Maximize:

\[ z = 3x_1 + x_2 + 2x_3 \]

Subject to:

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

**Basic variables** are on the left side and **non-basic variables** are on the right side of the equations

(the sets of basic and non-basic variables will change)
Convert the following standard form linear program into slack form:

Maximize:

\[ 2x_1 - 6x_3 \]

Subject to:

\[ x_1 + x_2 - x_3 \leq 7 \]
\[ -3x_1 + x_2 \leq -8 \]
\[ x_1 - 2x_2 - 2x_3 \leq 0 \]
\[ x_1, x_2, x_3 \geq 0 \]
Solution:

\[ z = 2x_1 - 6x_3 \]
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
\[ x_5 = -8 + 3x_1 - x_2 \]
\[ x_6 = -x_1 + 2x_2 + 2x_3 \]
\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]
Maximize:

\[ z = 3x_1 + x_2 + 2x_3 \]

Subject to:

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

3 equations and 6 unknowns (so infinite number of possibilities)
Maximize:

\[ z = 3x_1 + x_2 + 2x_3 \]

Subject to:

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

3 equations and 6 unknowns (so infinite number of possibilities)
Focus on basic solution: set all variables on the right-hand side set to 0.

Here: \( x_1 = 0, x_2 = 0, x_3 = 0 \), so \( x_4 = 30, x_5 = 24, x_6 = 36 \) and \( z = 0 \)
Goals

The simplex algorithm re-writes the set of equations and the objective function so that there’s different variables in the objective function. Re-writing the equations changes the basic solution (and therefore the objective function). Re-writing the equations does not change the system or underlying problem. Each iteration, increase the objective function by re-writing the equations.
Pivoting:

1. Select a non-basic variable ($x_e$, $e$ for entering) whose coefficient in the objective function is positive
2. Increase $x_e$ as much as possible
3. Switch $x_e$ with a basic variable, $x_l$ ($l$ for leaving)
Determining $x_e$

$$z = 3x_1 + x_2 + 2x_3$$

$x_1$ has the largest positive coefficient in the objective function

$x_e = x_1$
Determining $x_l$

To maximize the objective function using $x_1$:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$x_1 \leq 30$
Determining $x_l$

To maximize the objective function using $x_1$:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_1 \leq 30$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_1 \leq \frac{24}{2} = 12$$
Determining $x_l$

To maximize the objective function using $x_1$:

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

\[
x_1 \leq 30
\]
\[
x_1 \leq \frac{24}{2} = 12
\]
\[
x_1 \leq \frac{36}{4} = 9
\]
Determining $x_l$

To maximize the objective function using $x_1$:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$x_1 \leq 30$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$x_1 \leq \frac{24}{2} = 12$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$x_1 \leq \frac{36}{4} = 9$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Choose the tightest constraint, so $x_l = x_6$. 
First Pivot

\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

Solve for \( x_1 \):

\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]  \hspace{1cm} (23)

Now, re-write the other equations, substituting for \( x_1 \) using Eqn. 23:

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]  \hspace{1cm} (24)

\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]  \hspace{1cm} (25)

\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \]  \hspace{1cm} (26)

\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]  \hspace{1cm} (27)
Verify nothing changed

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]

\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]

\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \]

\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]

In the beginning, with
\[ x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36, z = 0. \] Is that still true?
Verify nothing changed

\[
z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}
\]

\[
x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}
\]

\[
x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}
\]

\[
x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\]

In the beginning, with \(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36, z = 0\). Is that still true? Yes!
The thing that changed was the basic solution (set non-basic variables to 0): \(x_1 = 9, x_2 = 0, x_3 = 0, x_4 = 21, x_5 = 6, x_6 = 0\), so \(z = 27\).
Second Iteration

Continue to increase objective function.
Determine $x_e$:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$

$x_3$ has the largest positive coefficient in the objective function
$x_e = x_3$
Determining $x_l$

To maximize the objective function using $x_3$:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$x_3 \leq \frac{9}{1.5} = 18$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

$$x_3 \leq \frac{21}{5.2} = \frac{42}{5} = 8.4$$

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

$$x_3 \leq \frac{6}{4} = \frac{3}{2} = 1.5$$

Choose the tightest constraint, so $x_l = x_5$. 
Second Pivot

\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]

Solve for \( x_3 \):

\[ x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{x_5}{4} + \frac{x_6}{8} \]  \hspace{1cm} (28)

Now, re-write the other equations, substituting for \( x_3 \) using Eqn. 28:

\[ z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \]  \hspace{1cm} (29)

\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]  \hspace{1cm} (30)

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]  \hspace{1cm} (31)

\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \]  \hspace{1cm} (32)
New basic solution (set non-basic variables to 0):

\[ x_1 = \frac{33}{4}, x_2 = 0, x_3 = \frac{3}{2}, x_4 = \frac{69}{4}, x_5 = 0, x_6 = 0, \text{ so} \]

\[ z = \frac{1111}{4} = 27.75. \]
Third Iteration

Continue to increase objective function. Determine $x_e$:

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$  \hspace{1cm} (33)

$x_2$ is the only way to increase the objective function

$x_e = x_2$
Determining $x_l$

To maximize the objective function using $x_2$:

\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]

\[ x_2 \leq \frac{\frac{33}{4}}{\frac{1}{16}} = 132 \]

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]

\[ x_2 \leq \frac{\frac{3}{2}}{\frac{3}{8}} = 4 \]

\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \]

\[ x_2 \leq \infty \]

Choose the tightest constraint, so $x_l = x_3$. 
Third Pivot

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]

Solve for \( x_2 \):

\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]

(34)

Now, re-write the other equations, substituting for \( x_2 \) using Eqn. 34:

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]

(35)

\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]

(36)

\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]

(37)

\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]

(38)
New basic solution (set non-basic variables to 0):
\[ x_1 = 8, \quad x_2 = 4, \quad x_3 = 0, \quad x_4 = 18, \quad x_5 = 0, \quad x_6 = 0, \] so \( z = 28. \)
Exercise

Apply the simplex algorithm to solve the following program:

Maximize:

\[ 18x_1 + 12.5x_2 \]

Subject to:

\[ x_1 + x_2 \leq 20 \]
\[ x_1 \leq 12 \]
\[ x_2 \leq 16 \]
\[ x_1, x_2 \geq 0 \]