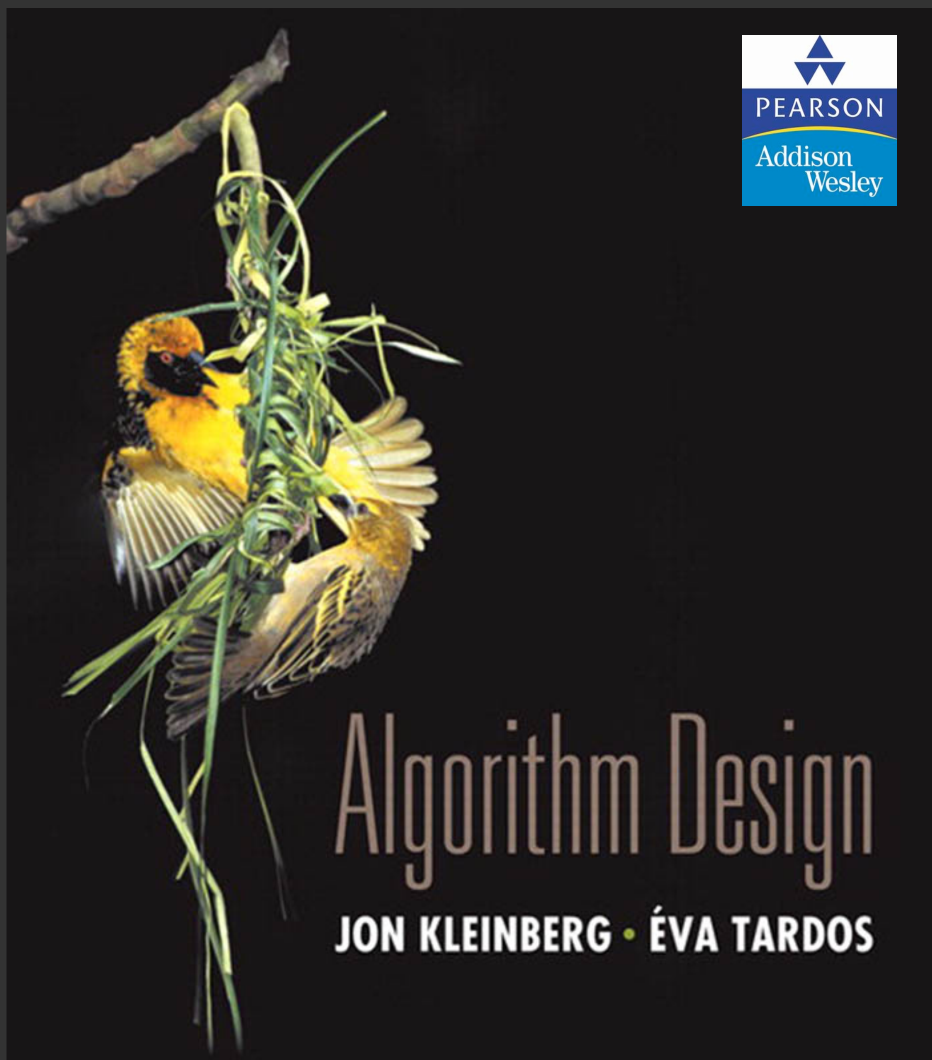


Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

# 8. Intractability I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

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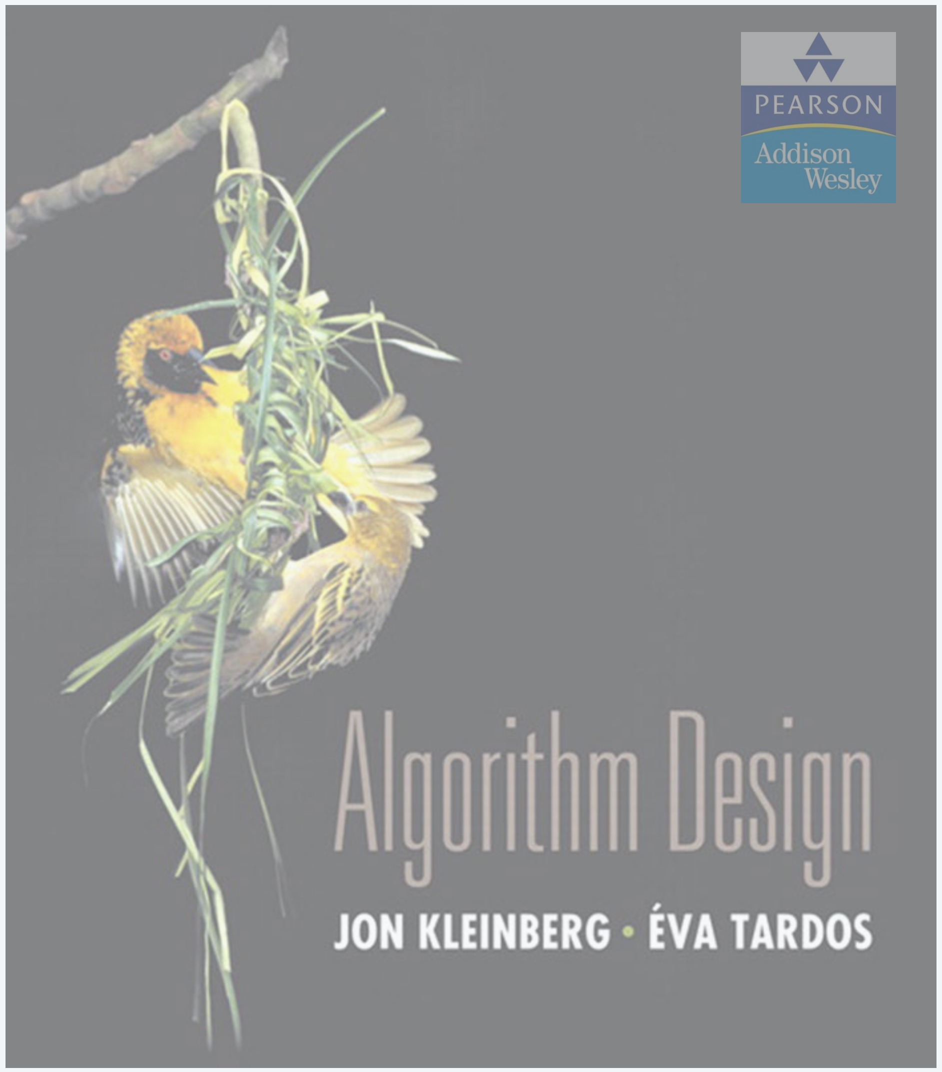



SECTION 8.1

# 8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

# Algorithm design patterns and antipatterns

Algorithm design patterns.

- Greedy.
- Divide and conquer.
- Dynamic programming.
- Reductions.
- Local search.
- Randomization.

Algorithm design antipatterns.

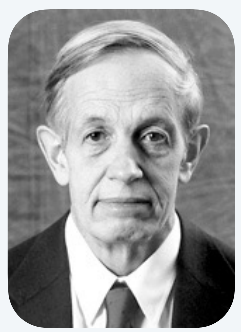
- **NP**-completeness. $O(n^k)$ algorithm unlikely.
- **PSPACE**-completeness. $O(n^k)$ certification algorithm unlikely.
- Undecidability. No algorithm possible.

# Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

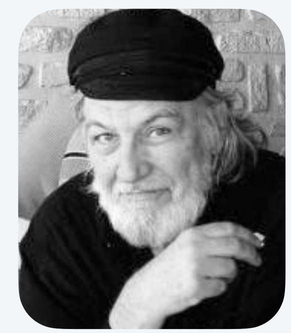
von Neumann (1953)



Nash (1955)

Gödel (1956)

Cobham (1964)



Edmonds (1965)

Rabin (1966)

Turing machine, word RAM, uniform circuits, …

Theory. Definition is broad and robust.

constants tend to be small, e.g., $3n^2$

Practice. Poly-time algorithms scale to huge problems.

# Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

| yes | probably no |
|---|---|
| shortest path | longest path |
| min cut | max cut |
| 2-satisfiability | 3-satisfiability |
| planar 4-colorability | planar 3-colorability |
| bipartite vertex cover | vertex cover |
| matching | 3d-matching |
| primality testing | factoring |
| linear programming | integer linear programming |

# Classify problems

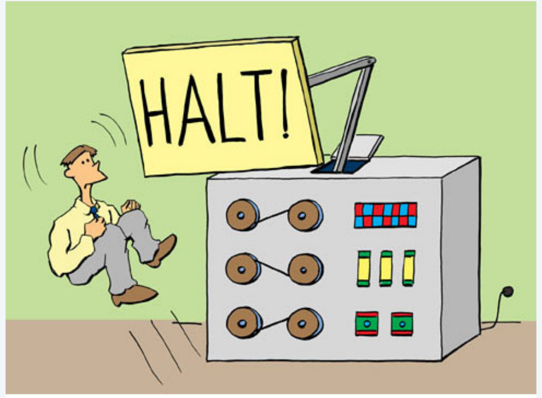
Desiderata. Classify problems according to those that can be solved in polynomial time and those that cannot.

Provably requires exponential time.

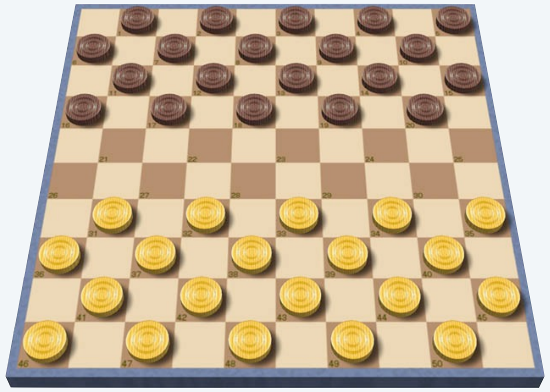
input size = $c + \log k$

- Given a constant-size program, does it halt in at most $k$ steps?
- Given a board position in an $n$-by-$n$ generalization of checkers, can black guarantee a win?

using forced capture rule




Alan designed the perfect computer



Frustrating news. Huge number of fundamental problems have defied classification for decades.

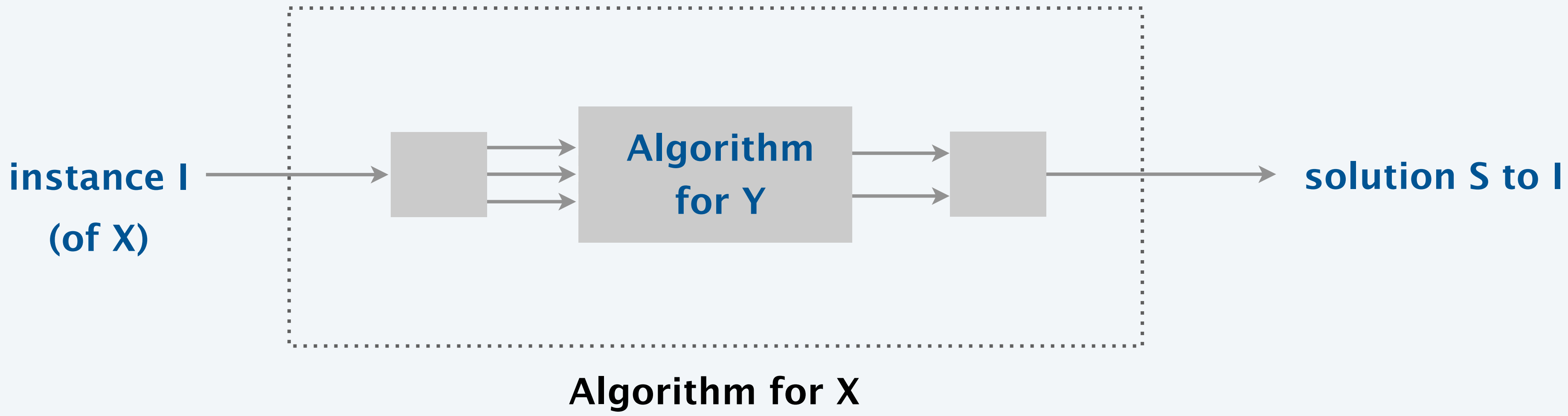
# Poly-time reductions

Desiderata′. Suppose we could solve problem $Y$ in polynomial time.
What else could we solve in polynomial time?

Reduction. Problem $X$ polynomial-time (Cook) reduces to problem $Y$ if arbitrary instances of problem $X$ can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem $Y$.

computational model supplemented by special piece of hardware that solves instances of $Y$ in a single step



# Poly-time reductions

Desiderata′. Suppose we could solve problem $Y$ in polynomial time.
What else could we solve in polynomial time?

Reduction. Problem $X$ polynomial-time (Cook) reduces to problem $Y$ if arbitrary instances of problem $X$ can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem $Y$.

Notation. $X \leq_P Y$.

Note. We pay for time to write down instances of $Y$ sent to oracle $\Rightarrow$ instances of $Y$ must be of polynomial size.

Novice mistake. Confusing $X \leq_P Y$ with $Y \leq_P X$.

# Intractability: quiz 1

**Suppose that $X \le_{\text{P}} Y$. Which of the following can we infer?**

**A.** If $X$ can be solved in polynomial time, then so can $Y$.

**B.** $X$ can be solved in poly time iff $Y$ can be solved in poly time.

**C.** If $X$ cannot be solved in polynomial time, then neither can $Y$.

**D.** If $Y$ cannot be solved in polynomial time, then neither can $X$.

# Intractability: quiz 2

**Which of the following poly-time reductions are known?**

**A.** FIND-MAX-FLOW $\leq_P$ FIND-MIN-CUT.

**B.** FIND-MIN-CUT $\leq_P$ FIND-MAX-FLOW.

**C.** Both A and B.

**D.** Neither A nor B.

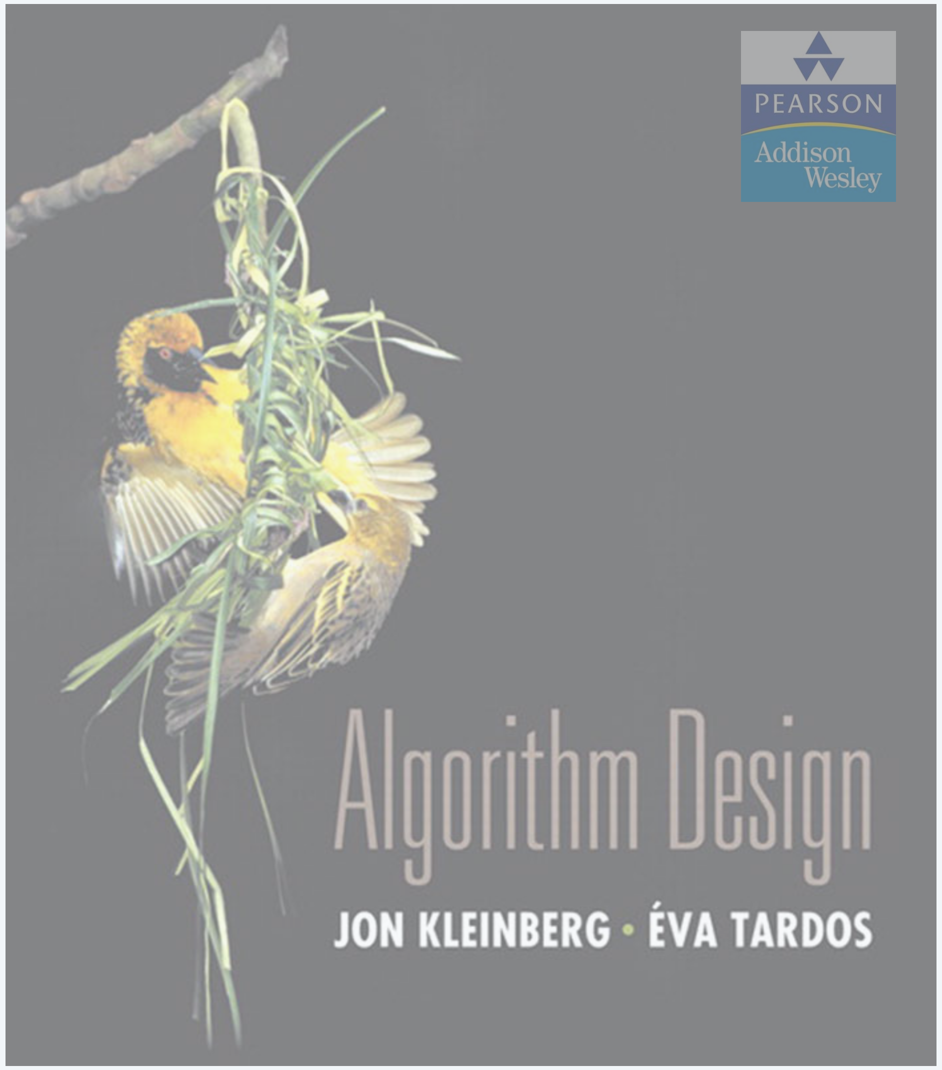
# Poly-time reductions

Design algorithms. If $X \leq_P Y$ and $Y$ can be solved in polynomial time, then $X$ can be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and $X$ cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

Establish equivalence. If both $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$. In this case, $X$ can be solved in polynomial time iff $Y$ can be.

Bottom line. Reductions classify problems according to relative difficulty.




SECTION 8.1

# 8. INTRACTABILITY I

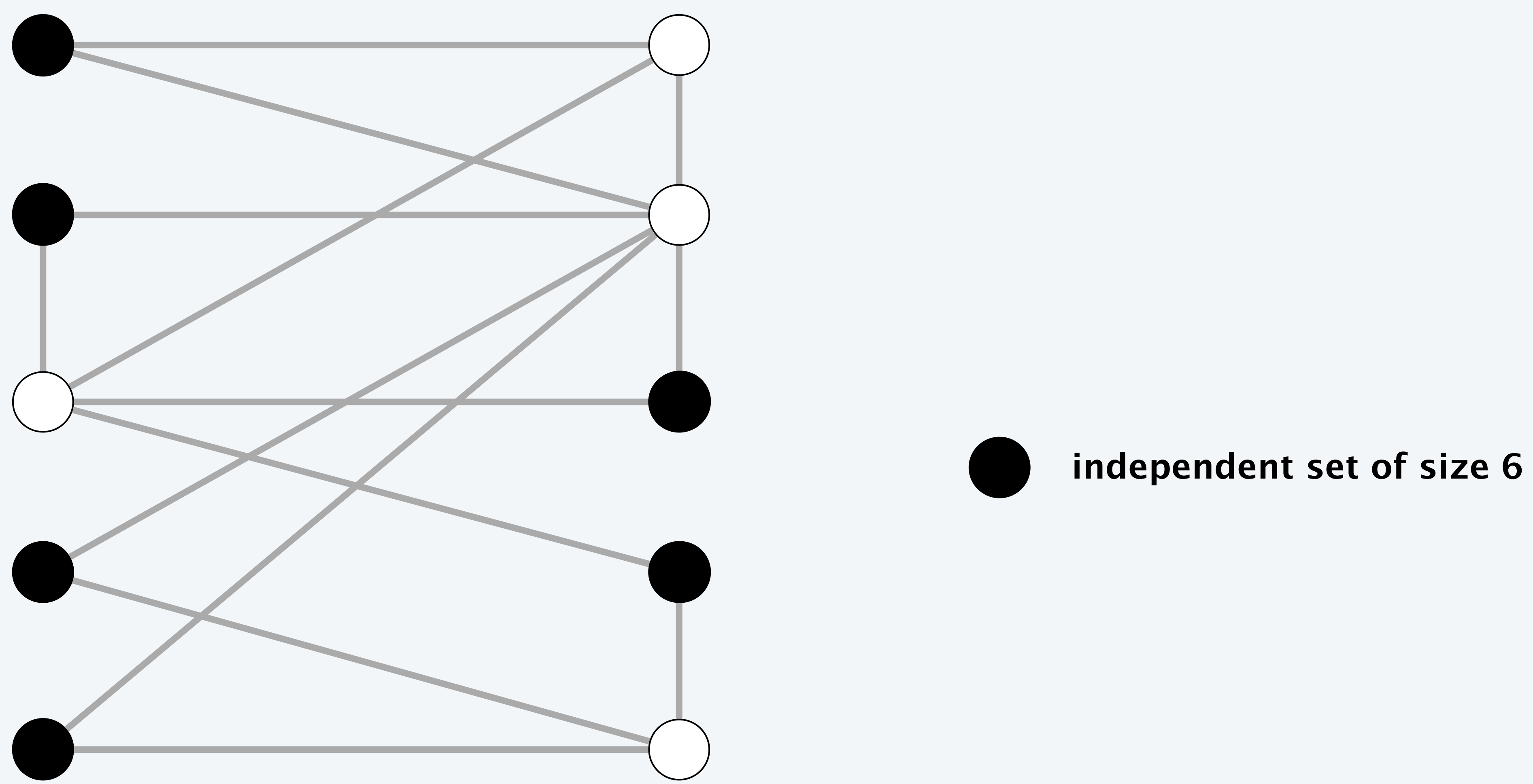
- poly-time reductions
- *packing and covering problems*
- constraint satisfaction problems
- sequencing problems
- partitioning problems
- graph coloring
- numerical problems

# Independent set

INDEPENDENT-SET. Given a graph $G = (V, E)$ and an integer $k$, is there a subset of $k$ (or more) vertices such that no two are adjacent?

Ex. Is there an independent set of size $\geq 6$ ?

Ex. Is there an independent set of size $\geq 7$ ?

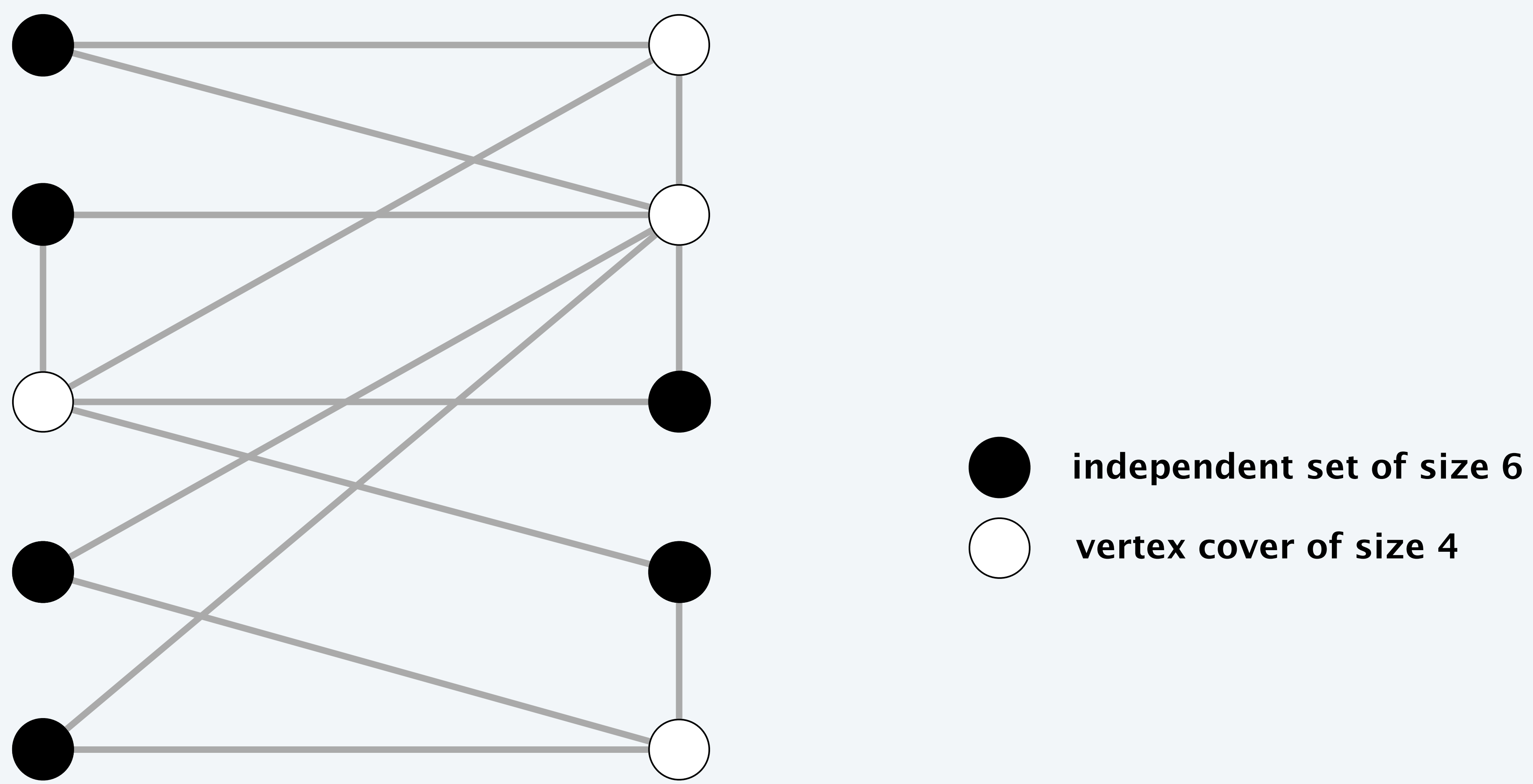


# Vertex cover

VERTEX-COVER. Given a graph $G = (V, E)$ and an integer $k$, is there a subset of $k$ (or fewer) vertices such that each edge is incident to at least one vertex in the subset?

Ex. Is there a vertex cover of size $\leq 4$ ?

Ex. Is there a vertex cover of size $\leq 3$ ?



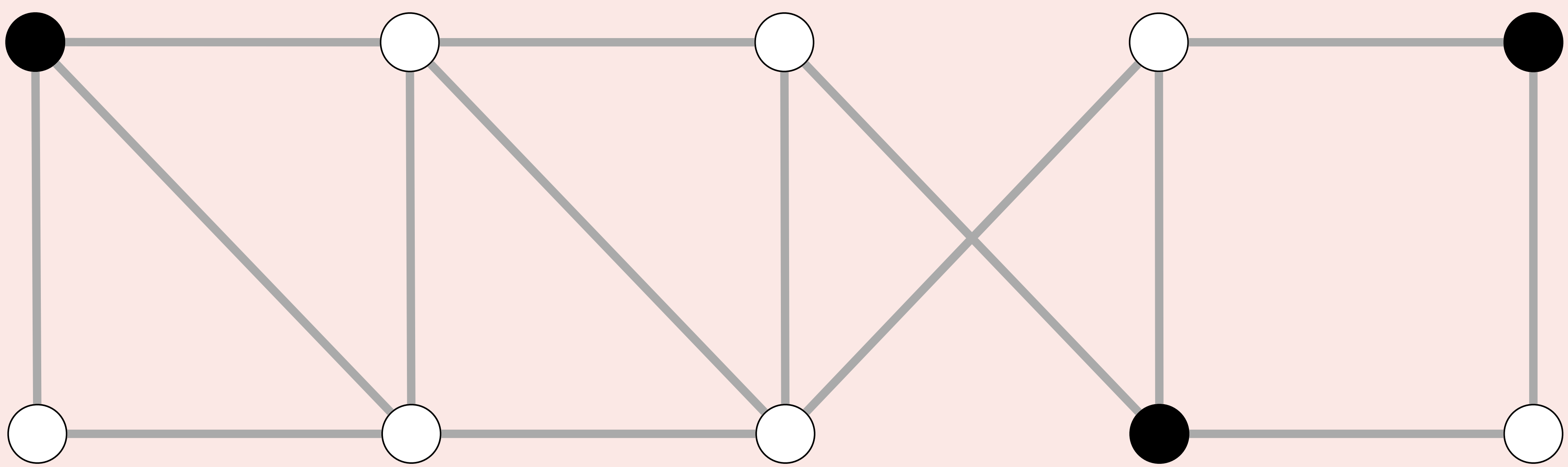
**Consider the following graph G. Which are true?**

**A.** The white vertices are a vertex cover of size 7.

**B.** The black vertices are an independent set of size 3.

**C.** Both A and B.

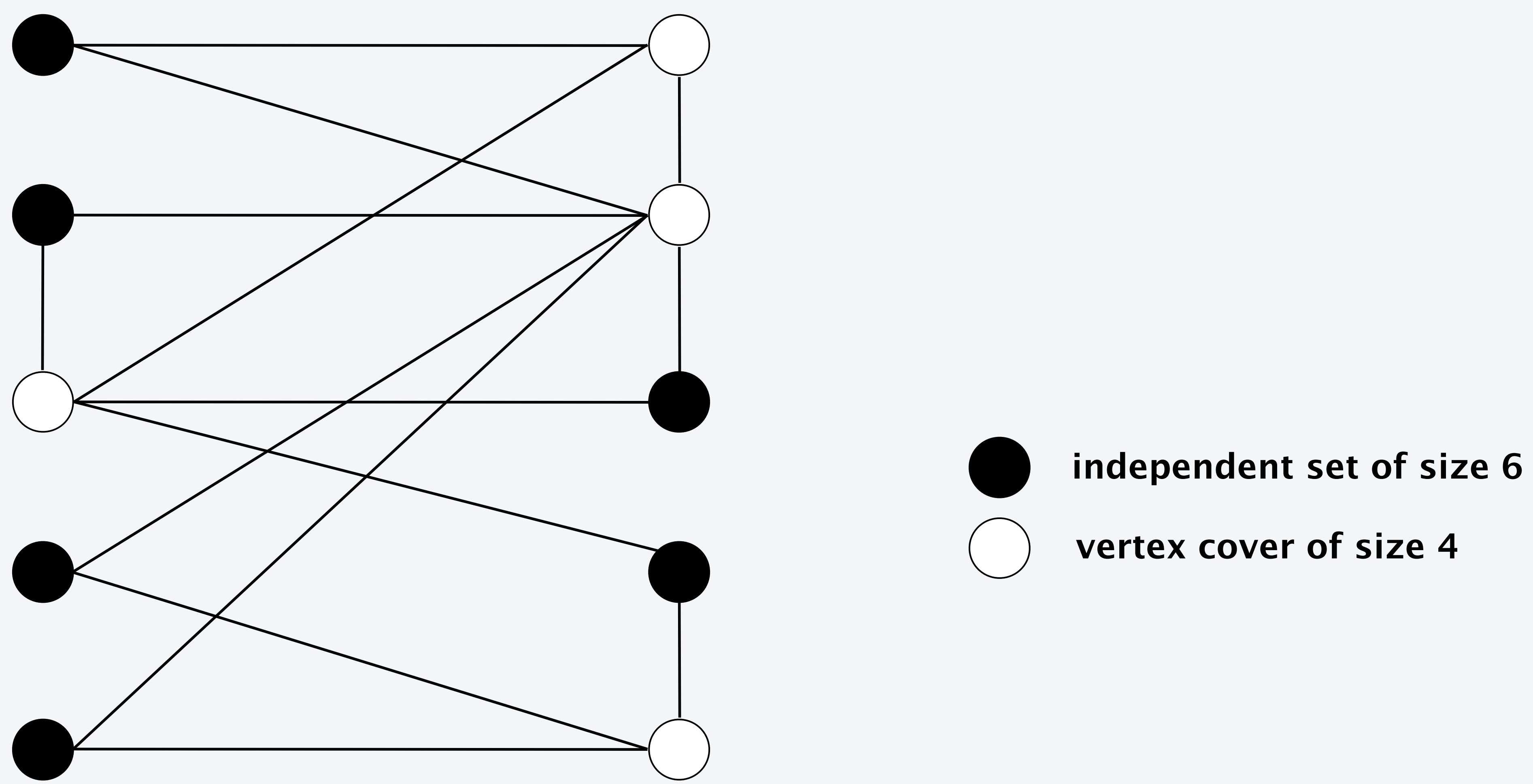
**D.** Neither A nor B.



# Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show $S$ is an independent set of size $k$ iff $V - S$ is a vertex cover of size $n - k$.



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Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show $S$ is an independent set of size $k$ iff $V - S$ is a vertex cover of size $n - k$.

$\Rightarrow$

- Let $S$ be any independent set of size $k$.
- $V - S$ is of size $n - k$.
- Consider an arbitrary edge $(u, v) \in E$.
- $S$ independent $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
  $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
- Thus, $V - S$ covers $(u, v)$. ▪

# Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show $S$ is an independent set of size $k$ iff $V - S$ is a vertex cover of size $n - k$.

$\Leftarrow$

- Let $V - S$ be any vertex cover of size $n - k$.
- $S$ is of size $k$.
- Consider an arbitrary edge $(u, v) \in E$.
- $V - S$ is a vertex cover $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
  $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
- Thus, $S$ is an independent set. ▪

# Set cover

SET-COVER. Given a set $U$ of elements, a collection $S$ of subsets of $U$, and an integer $k$, are there $\leq k$ of these subsets whose union is equal to $U$ ?

Sample application.

- $m$ available pieces of software.
- Set $U$ of $n$ capabilities that we would like our system to have.
- The $i^{th}$ piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all $n$ capabilities using fewest pieces of software.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

$k = 2$

a set cover instance

**Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?**

**A.** 1

**B.** 2

**C.** 3

**D.** None of the above.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 1, 4, 6 \}$ $\quad S_b = \{ 1, 6, 7 \}$

$S_c = \{ 1, 2, 3, 6 \}$ $\quad S_d = \{ 1, 3, 5, 7 \}$

$S_e = \{ 2, 6, 7 \}$ $\quad S_f = \{ 3, 4, 5 \}$

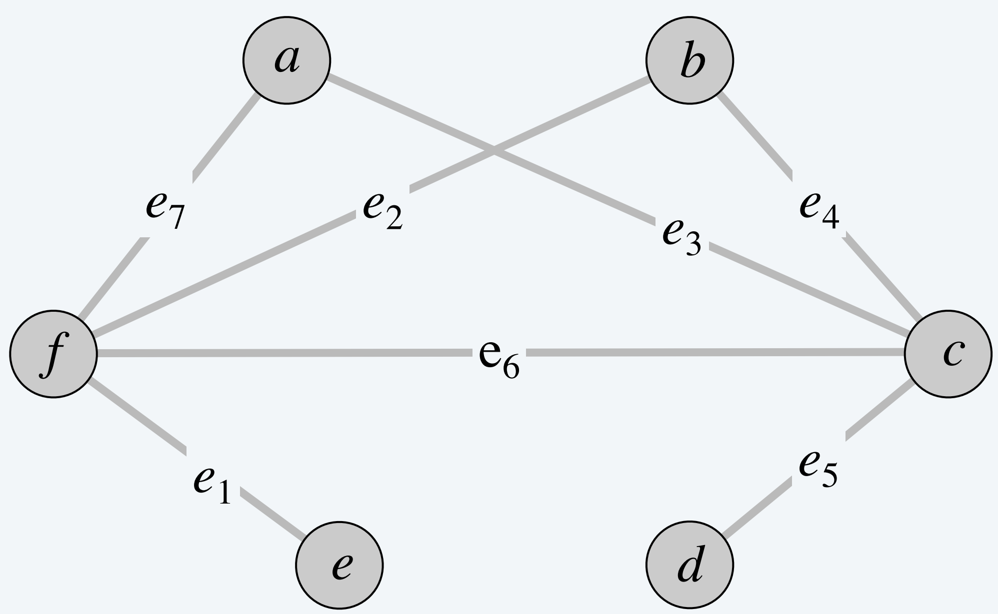
# Vertex cover reduces to set cover

Theorem. VERTEX-COVER $\leq_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$ and $k$, we construct a SET-COVER instance $(U, S, k)$ that has a set cover of size $k$ iff $G$ has a vertex cover of size $k$.

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V : S_v = \{e \in E : e \text{ incident to } v\}$.




vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

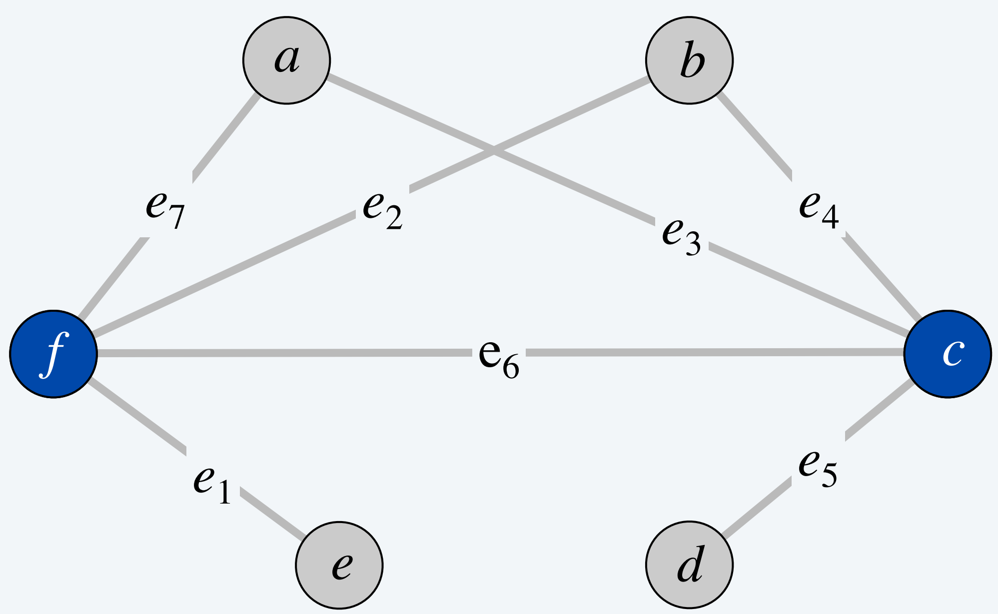
# Vertex cover reduces to set cover

**Lemma.** $G = (V, E)$ contains a vertex cover of size $k$ iff $(U, S, k)$ contains a set cover of size $k$.

**Pf.** $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size $k$ in $G$.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size $k$. ▪

"yes" instances of VERTEX-COVER are solved correctly




vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

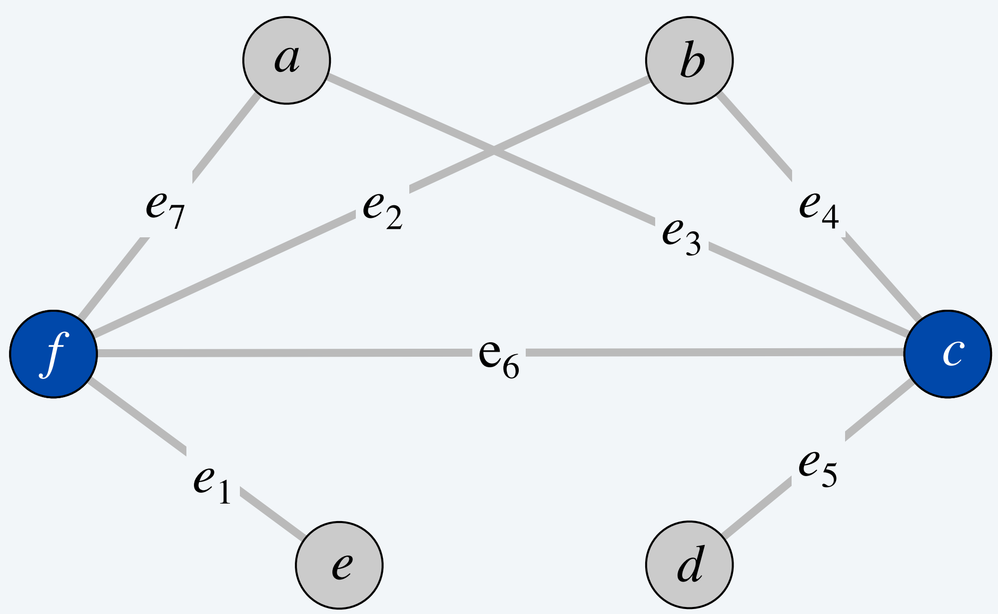
# Vertex cover reduces to set cover

**Lemma.** $G = (V, E)$ contains a vertex cover of size $k$ iff $(U, S, k)$ contains a set cover of size $k$.

**Pf.** $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size $k$ in $(U, S, k)$.

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size $k$ in $G$. ▪

"no" instances of VERTEX-COVER are solved correctly




**vertex cover instance**
**(k = 2)**

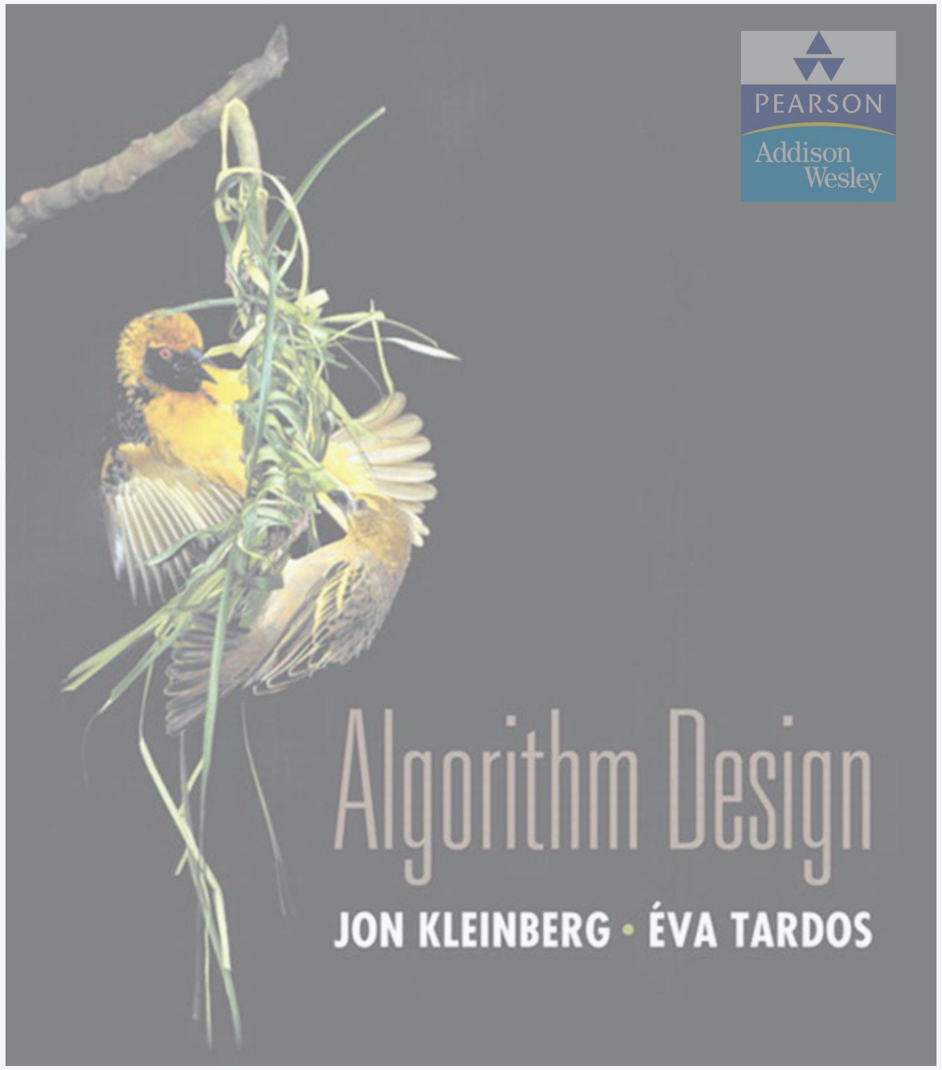
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$\boxed{S_c = \{ 3, 4, 5, 6 \}}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $\boxed{S_f = \{ 1, 2, 6, 7 \}}$

**set cover instance**
**(k = 2)**




SECTION 8.2

# 8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

# Satisfiability

Literal. A Boolean variable or its negation. $x_i$ or $\overline{x_i}$

Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula $\Phi$ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula $\Phi$, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals
(and each literal corresponds to a different variable).

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

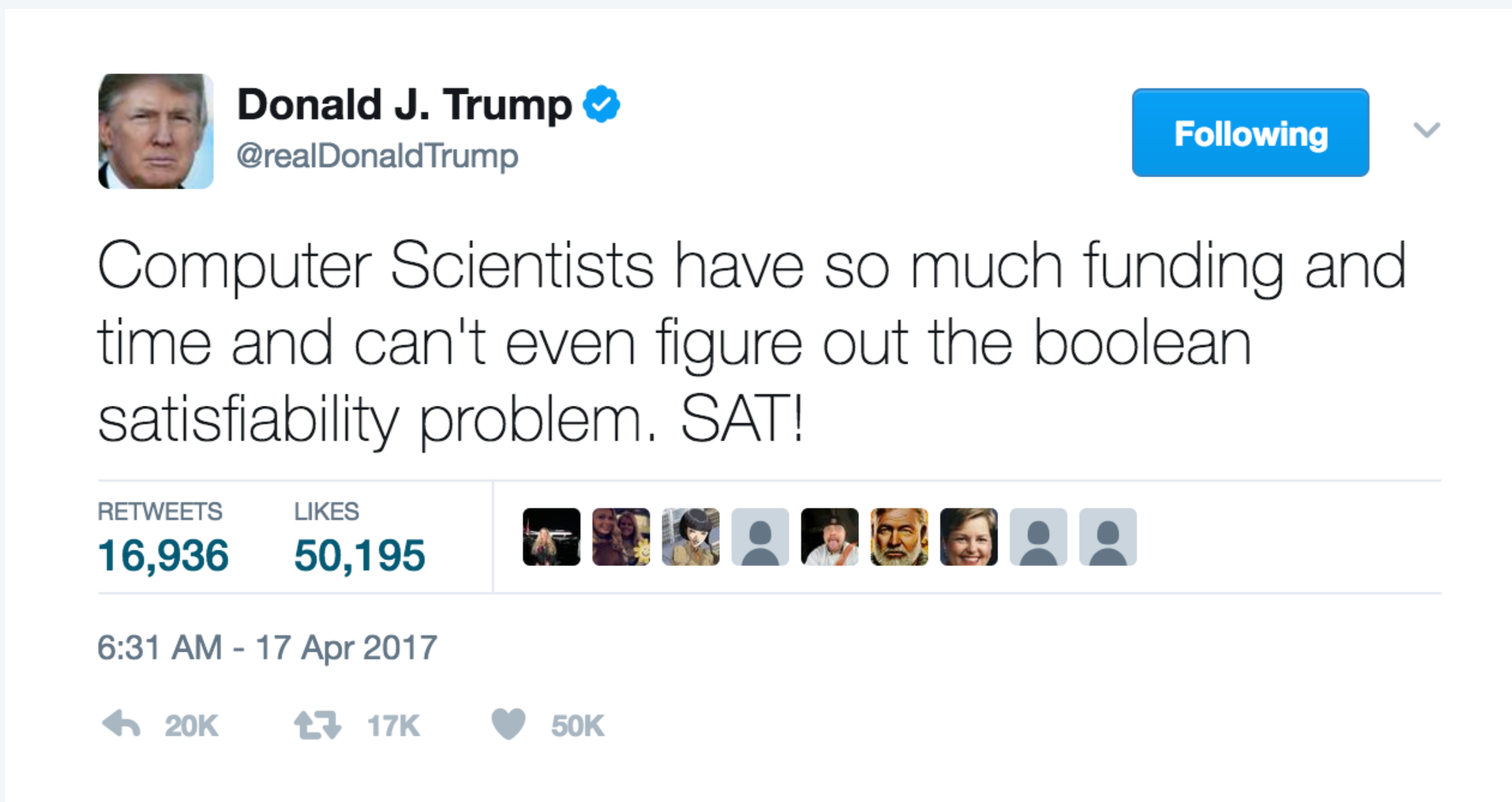
**yes instance: $x_1$ = true, $x_2$ = true, $x_3$ = false, $x_4$ = false**

Key application. Electronic design automation (EDA).

# Satisfiability is hard

Scientific hypothesis. There does not exist a poly-time algorithm for 3-SAT.

P vs. NP. This hypothesis is equivalent to **P ≠ NP** conjecture.




https://www.facebook.com/pg/npcompleteteens

# 3-satisfiability reduces to independent set

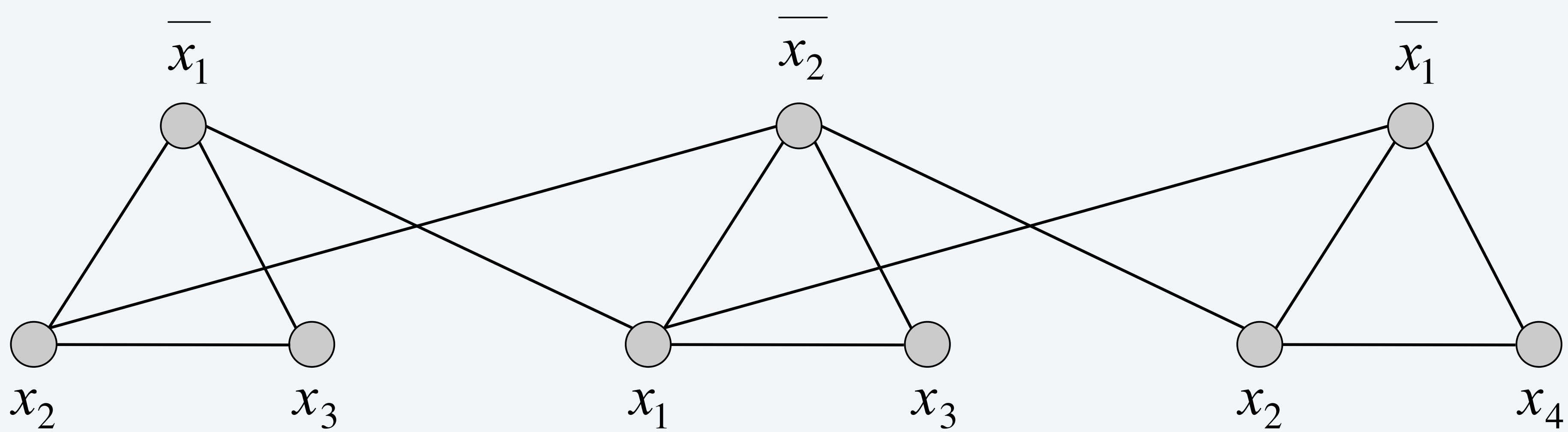
Theorem. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance $(G, k)$ of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff $\Phi$ is satisfiable.

Construction.

- $G$ contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G




k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

# 3-satisfiability reduces to independent set

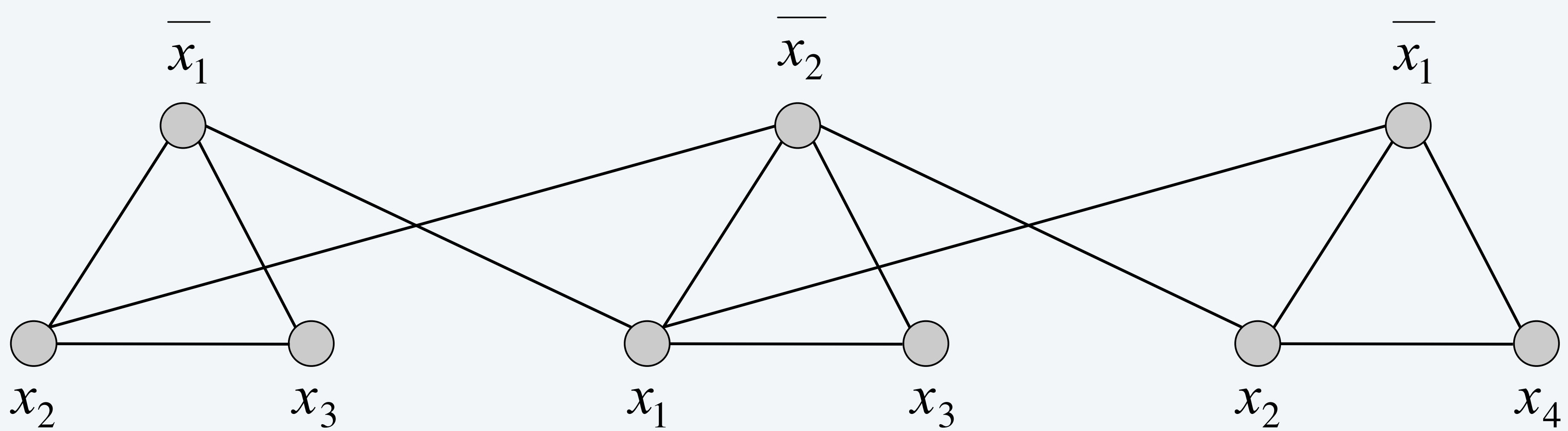
Lemma. $\Phi$ is satisfiable iff $G$ contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for $\Phi$.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G




k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

# 3-satisfiability reduces to independent set

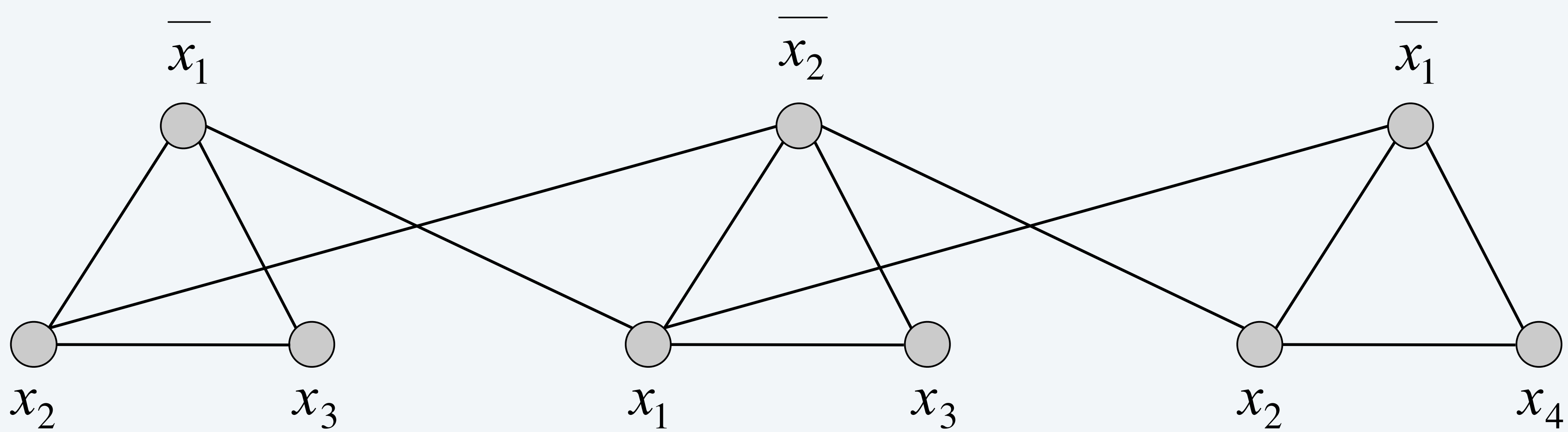
Lemma. $\Phi$ is satisfiable iff $G$ contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let $S$ be independent set of size $k$.

- $S$ must contain exactly one node in each triangle.
- Set these literals to *true* (and remaining literals consistently).
- All clauses in $\Phi$ are satisfied. ▪

"no" instances of 3-SAT are solved correctly

G




k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

# Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_{\text{P}}$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\le_{\text{P}}$ SET-COVER.
- Encoding with gadgets: 3-SAT $\le_{\text{P}}$ INDEPENDENT-SET.

Transitivity. If $X \le_{\text{P}} Y$ and $Y \le_{\text{P}} Z$, then $X \le_{\text{P}} Z$.

Pf idea. Compose the two algorithms.

Ex. 3-SAT $\le_{\text{P}}$ INDEPENDENT-SET $\le_{\text{P}}$ VERTEX-COVER $\le_{\text{P}}$ SET-COVER.

# Decision, search, and optimization problems

Decision problem. Does there exist a vertex cover of size $\leq k$ ?

Search problem. Find a vertex cover of size $\leq k$.

Optimization problem. Find a vertex cover of minimum size.

Goal. Show that all three problems poly-time reduce to one another.

# SEARCH PROBLEMS VS. DECISION PROBLEMS

VERTEX-COVER. Does there exist a vertex cover of size $\leq k$ ?

FIND-VERTEX-COVER. Find a vertex cover of size $\leq k$.

Theorem. VERTEX-COVER $\equiv_P$ FIND-VERTEX-COVER.

Pf. $\leq_P$ Decision problem is a special case of search problem. ▪

Pf. $\geq_P$

To find a vertex cover of size $\leq k$ :

- Determine if there exists a vertex cover of size $\leq k$.
- Find a vertex $v$ such that $G - \{ v \}$ has a vertex cover of size $\leq k - 1$. (any vertex in any vertex cover of size $\leq k$ will have this property)
- Include $v$ in the vertex cover.
- Recursively find a vertex cover of size $\leq k - 1$ in $G - \{ v \}$. ▪

delete $v$ and all incident edges

# OPTIMIZATION PROBLEMS VS. SEARCH PROBLEMS

FIND-VERTEX-COVER. Find a vertex cover of size $\leq k$.

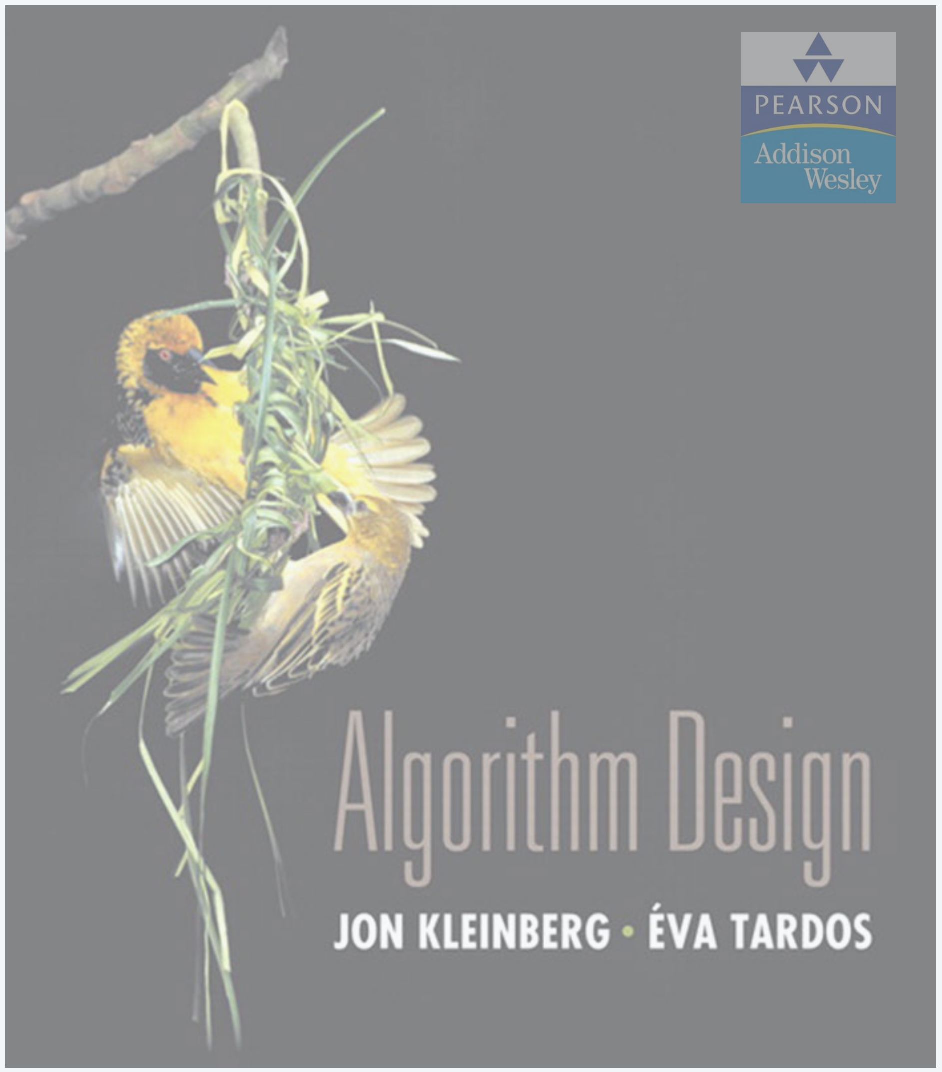
FIND-MIN-VERTEX-COVER. Find a vertex cover of minimum size.

Theorem. FIND-VERTEX-COVER $\equiv_P$ FIND-MIN-VERTEX-COVER.

Pf. $\leq_P$ Search problem is a special case of optimization problem. ▪

Pf. $\geq_P$ To find vertex cover of minimum size:

- Binary search (or linear search) for size $k^*$ of min vertex cover.
- Solve search problem for given $k^*$. ▪



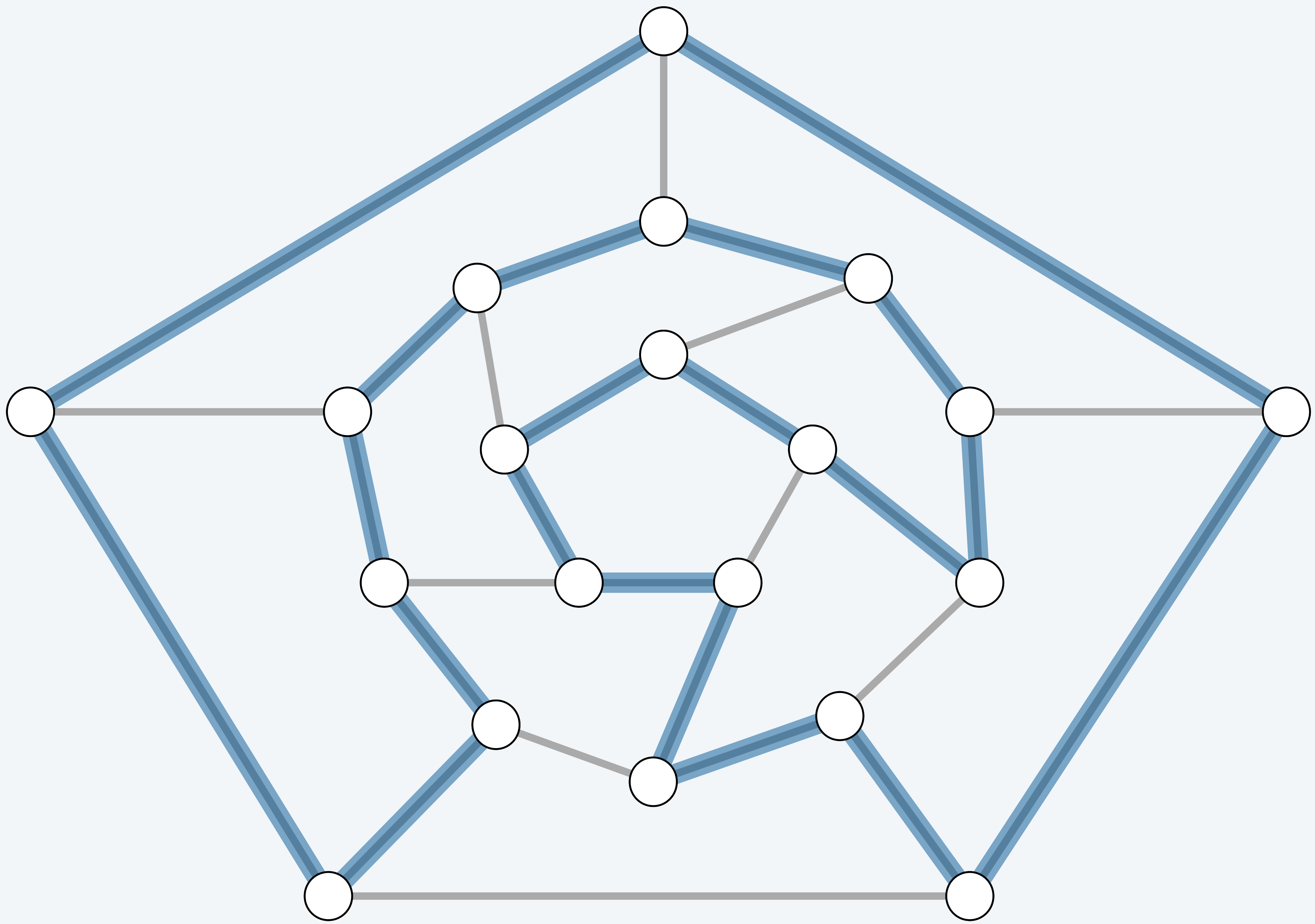

SECTION 8.5

# 8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- ***sequencing problems***
- *partitioning problems*
- *graph coloring*
- *numerical problems*

# Hamilton cycle

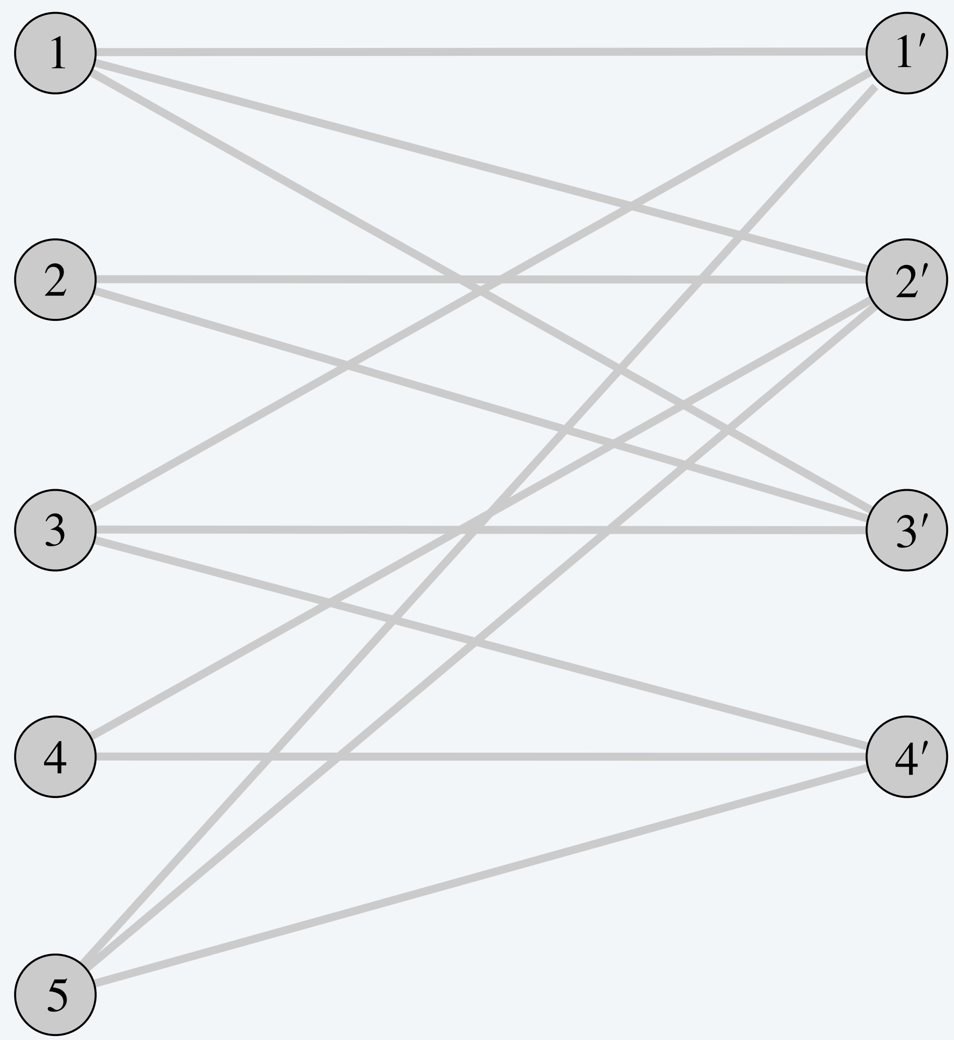
HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle $\Gamma$ that visits every node exactly once?



yes

# Hamilton cycle

HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle $\Gamma$ that visits every node exactly once?



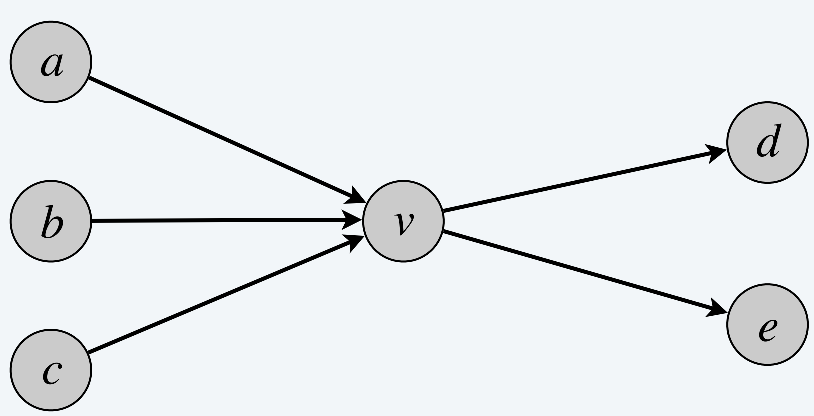

**no**

# Directed Hamilton cycle reduces to Hamilton cycle

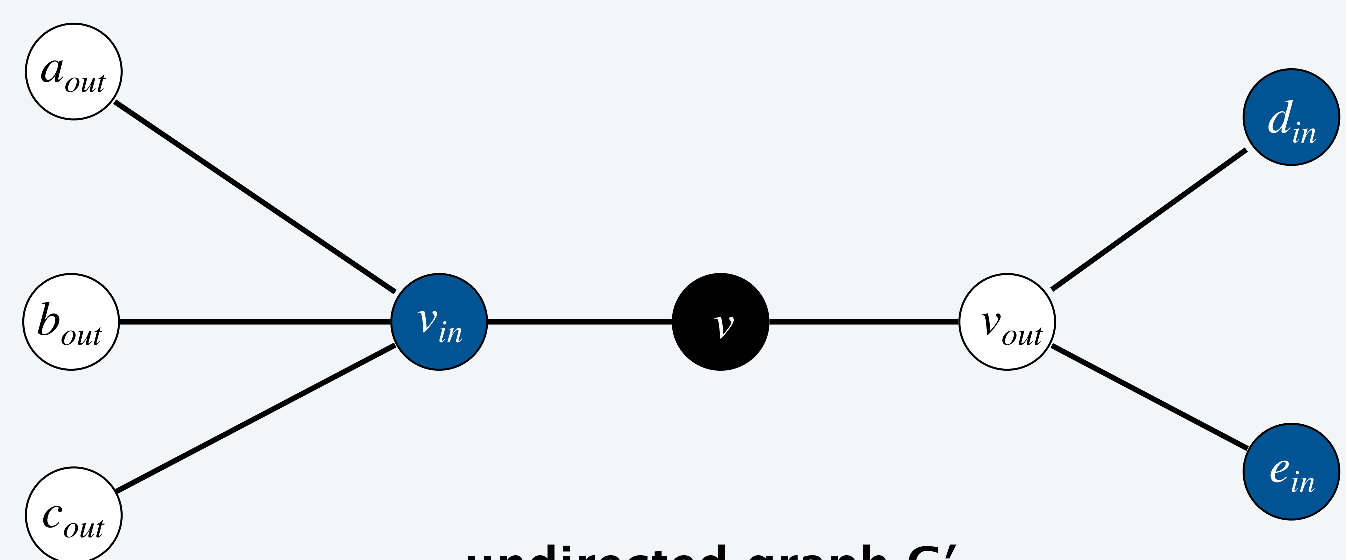
DIRECTED-HAMILTON-CYCLE. Given a directed graph $G = (V, E)$, does there exist a directed cycle $\Gamma$ that visits every node exactly once?

Theorem. DIRECTED-HAMILTON-CYCLE $\leq_P$ HAMILTON-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct a graph $G'$ with $3n$ nodes.




**directed graph G**




**undirected graph G′**

# Directed Hamilton cycle reduces to Hamilton cycle

**Lemma.** $G$ has a directed Hamilton cycle iff $G'$ has a Hamilton cycle.

**Pf.** $\Rightarrow$

- Suppose $G$ has a directed Hamilton cycle $\Gamma$.
- Then $G'$ has an undirected Hamilton cycle (same order). ▪

**Pf.** $\Leftarrow$

- Suppose $G'$ has an undirected Hamilton cycle $\Gamma'$.
- $\Gamma'$ must visit nodes in $G'$ using one of following two orders:

$$\ldots, \mathit{black}, \mathit{white}, \mathit{blue}, \mathit{black}, \mathit{white}, \mathit{blue}, \mathit{black}, \mathit{white}, \mathit{blue}, \ldots$$

$$\ldots, \mathit{black}, \mathit{blue}, \mathit{white}, \mathit{black}, \mathit{blue}, \mathit{white}, \mathit{black}, \mathit{blue}, \mathit{white}, \ldots$$

- Black nodes in $\Gamma'$ comprise either a directed Hamilton cycle $\Gamma$ in $G$, or reverse of one. ▪

# 3-satisfiability reduces to directed Hamilton cycle

Theorem. 3-SAT $\le_P$ DIRECTED-HAMILTON-CYCLE.

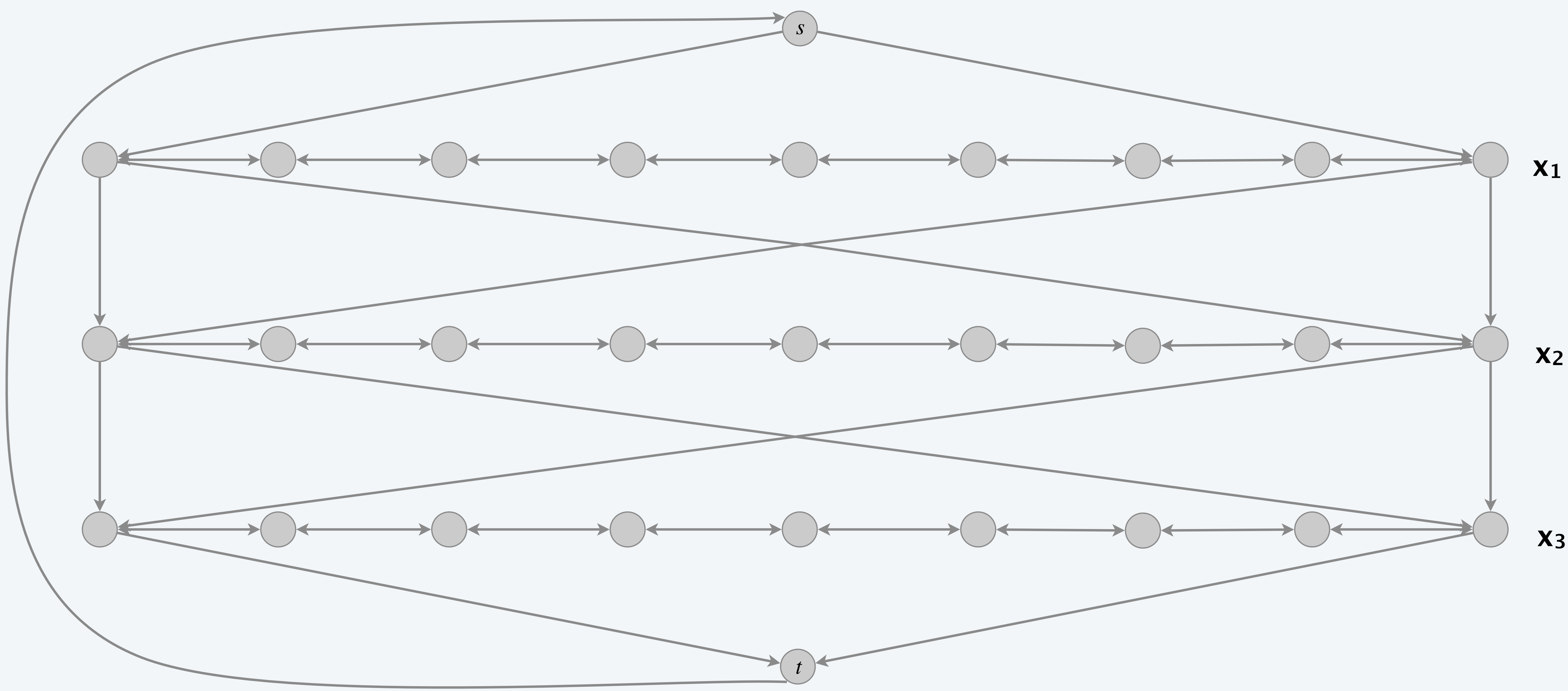
Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance $G$ of DIRECTED-HAMILTON-CYCLE that has a Hamilton cycle iff $\Phi$ is satisfiable.

Construction overview. Let $n$ denote the number of variables in $\Phi$. We will construct a graph $G$ that has $2^n$ Hamilton cycles, with each cycle corresponding to one of the $2^n$ possible truth assignments.

# 3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- Construct $G$ to have $2^n$ Hamilton cycles.
- Intuition: traverse path $i$ from left to right $\Leftrightarrow$ set variable $x_i = true$.



# Intractability: quiz 5

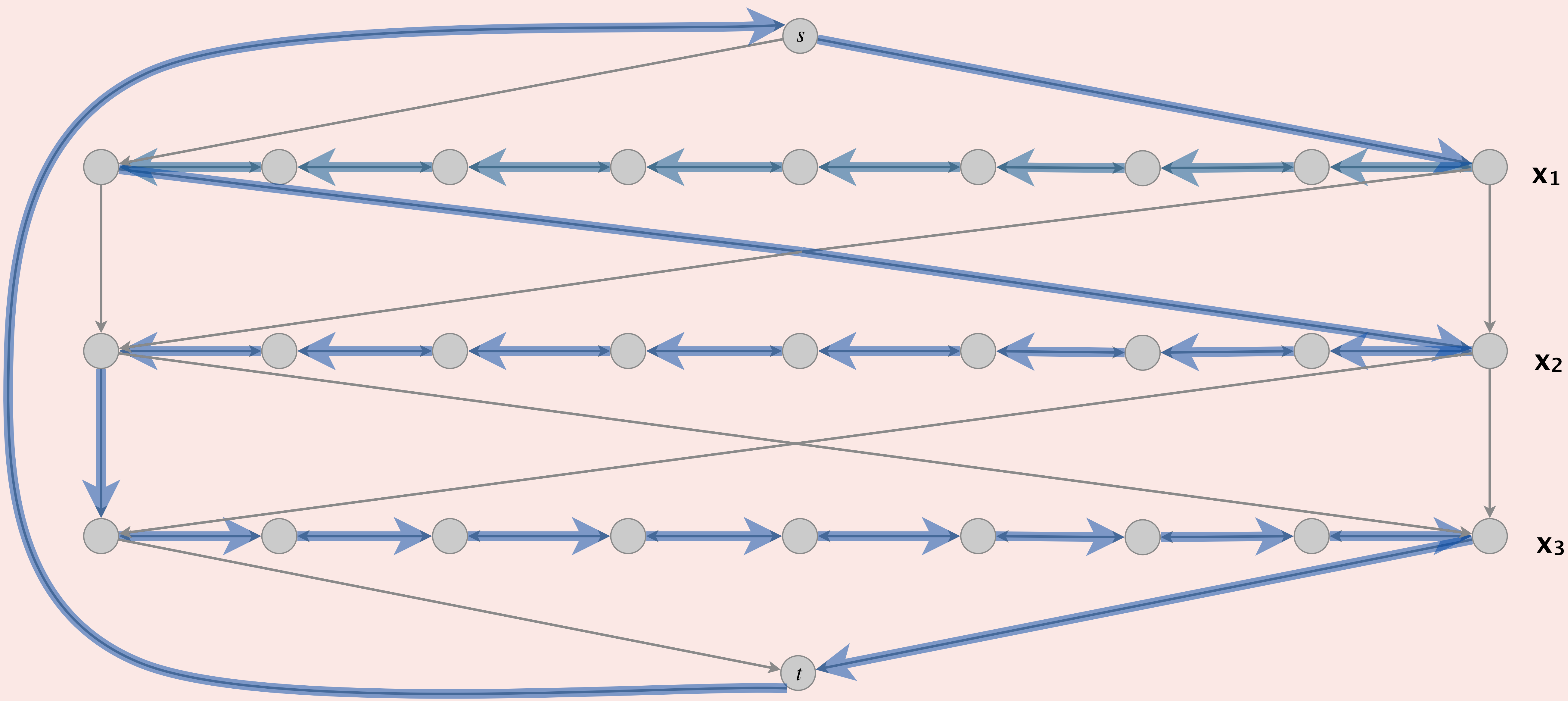
**Which is truth assignment corresponding to Hamilton cycle below?**

**A.** $x_1 = true, x_2 = true, x_3 = true$

**B.** $x_1 = true, x_2 = true, x_3 = false$

**C.** $x_1 = false, x_2 = false, x_3 = true$

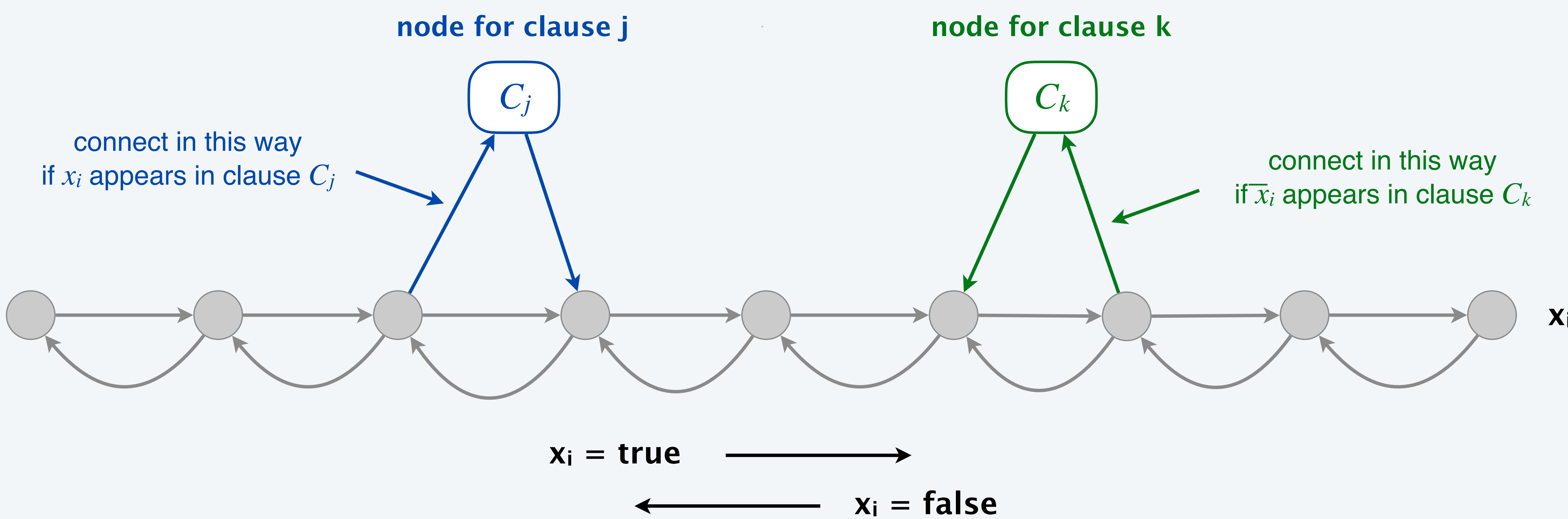
**D.** $x_1 = false, x_2 = false, x_3 = false$



# 3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

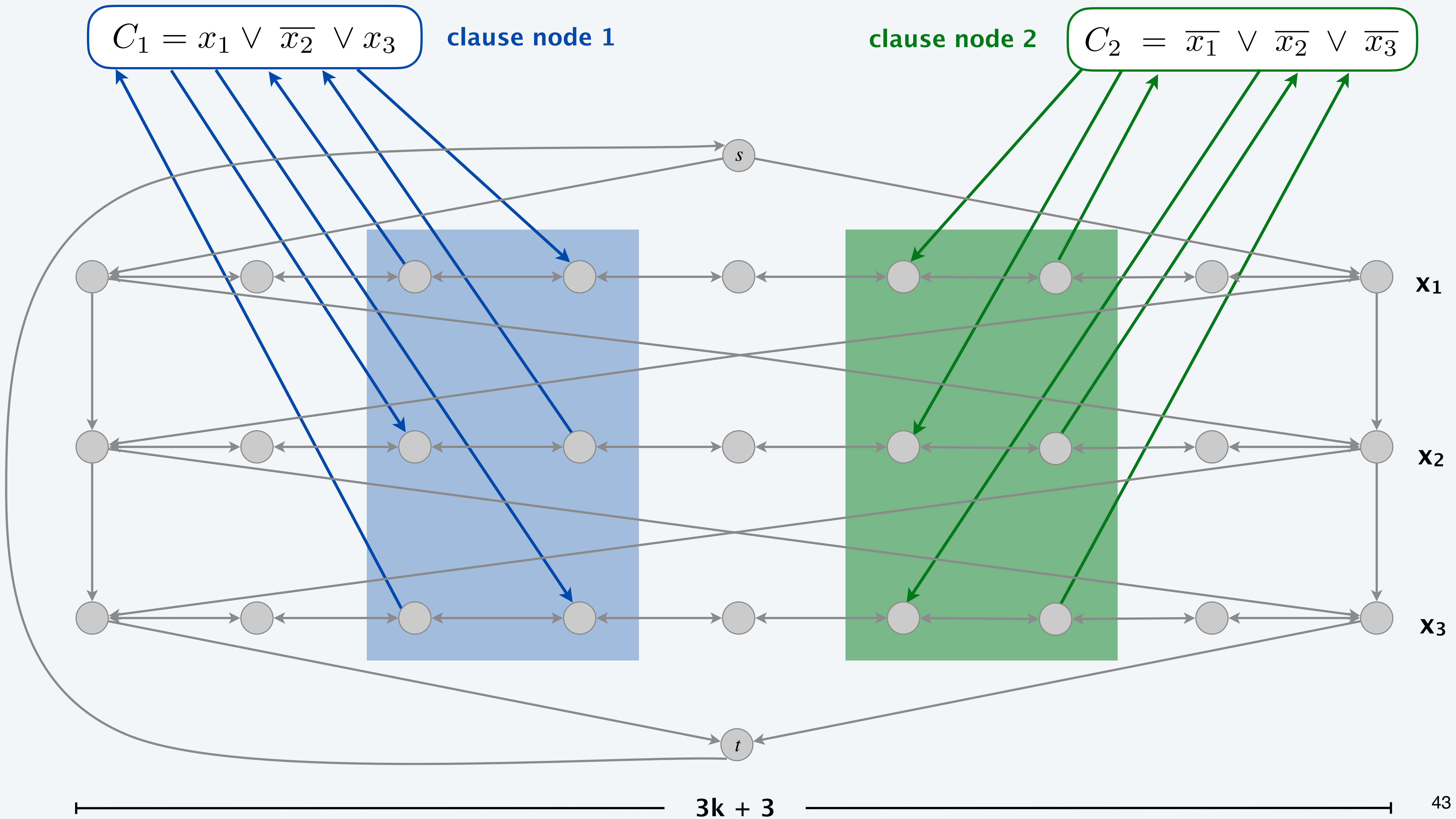
- For each clause: add a node and 2 edges per literal.



# 3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- For each clause: add a node and 2 edges per literal.



# 3-satisfiability reduces to directed Hamilton cycle

Lemma. $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance $\Phi$ has satisfying assignment $x^*$.
- Then, define Hamilton cycle $\Gamma$ in $G$ as follows:
  - if $x_i^* = true$, traverse row $i$ from left to right
  - if $x_i^* = false$, traverse row $i$ from right to left
  - for each clause $C_j$, there will be at least one row $i$ in which we are going in "correct" direction to splice clause node $C_j$ into cycle (and we splice in $C_j$ exactly once) ▪

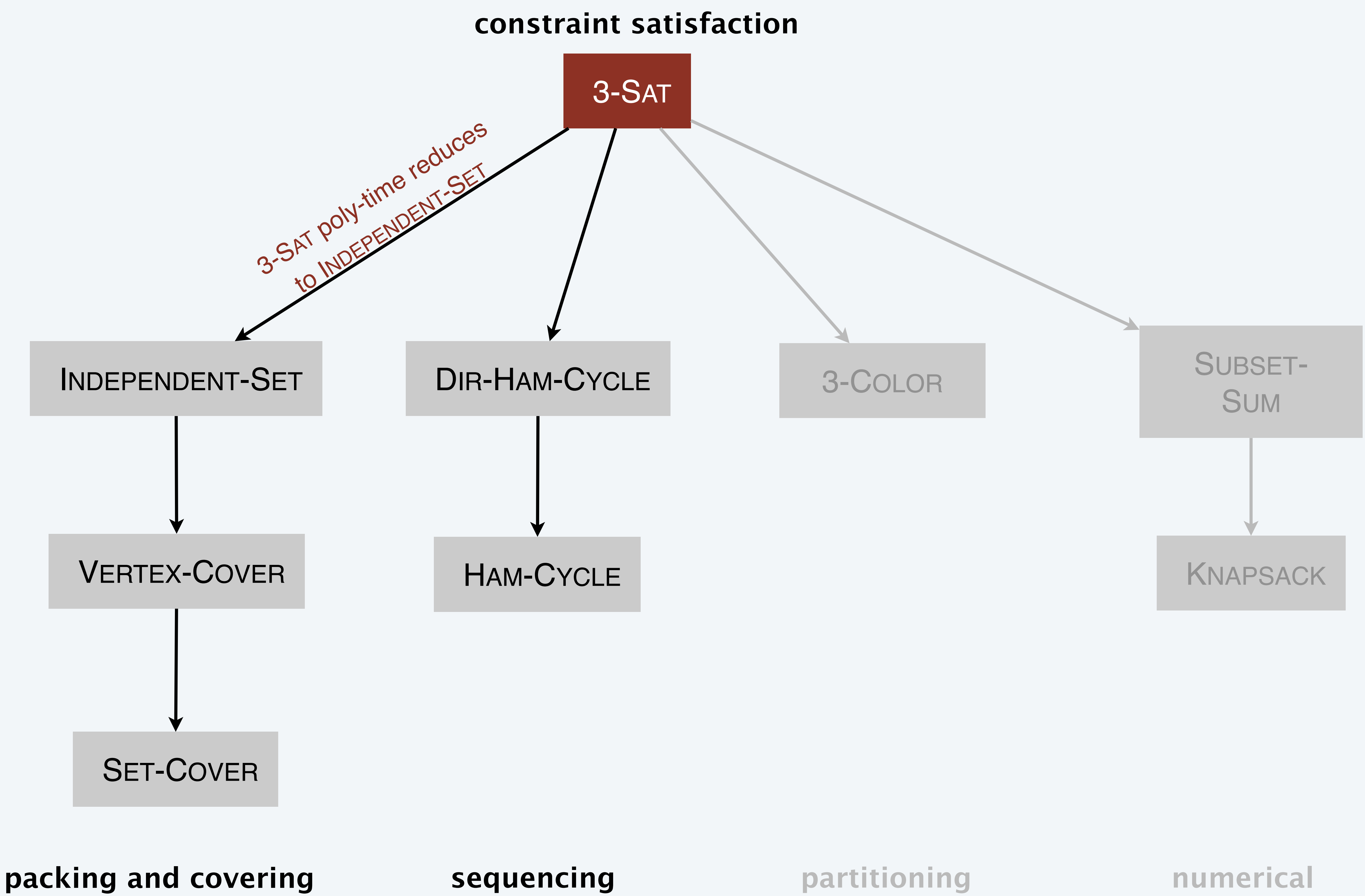
# 3-satisfiability reduces to directed Hamilton cycle

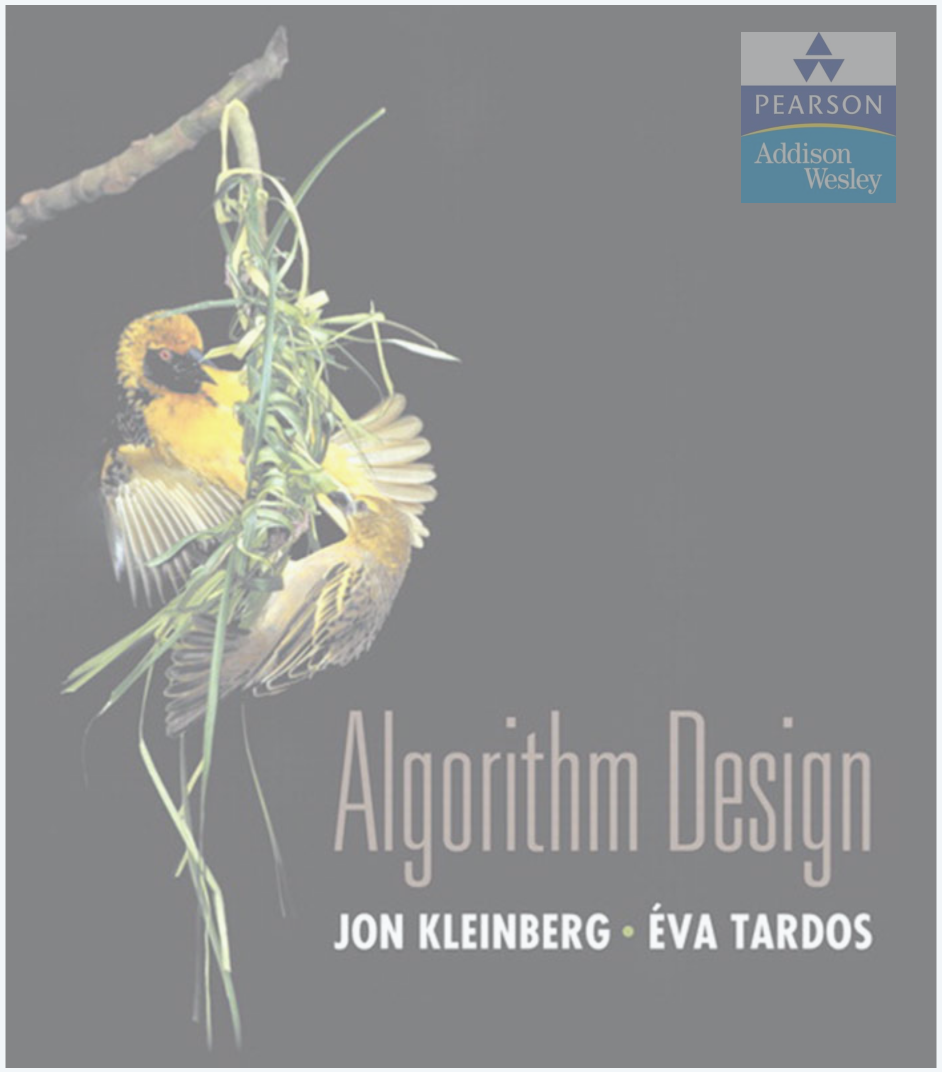
Lemma. $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf. $\Leftarrow$

- Suppose $G$ has a Hamilton cycle $\Gamma$.
- If $\Gamma$ enters clause node $C_j$, it must depart on mate edge.
  - nodes immediately before and after $C_j$ are connected by an edge $e \in E$
  - removing $C_j$ from cycle, and replacing it with edge $e$ yields Hamilton cycle on $G - \{ C_j \}$
- Continuing in this way, we are left with a Hamilton cycle $\Gamma'$ in $G - \{ C_1, C_2, \ldots, C_k \}$.
- Set $x_i^* = true$ if $\Gamma'$ traverses row $i$ left-to-right; otherwise, set $x_i^* = false$.
- traversed in "correct" direction, and each clause is satisfied. ▪

# Poly-time reductions






SECTION 8.6

# 8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

# 3-dimensional matching

3D-MATCHING. Given $n$ instructors, $n$ courses, and $n$ times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

| instructor | course | time |
|---|---|---|
| Wayne | COS 226 | TTh 11–12:20 |
| Wayne | COS 423 | MW 11–12:20 |
| Wayne | COS 423 | TTh 11–12:20 |
| Tardos | COS 423 | TTh 3–4:20 |
| Tardos | COS 523 | TTh 3–4:20 |
| Kleinberg | COS 226 | TTh 3–4:20 |
| Kleinberg | COS 226 | MW 11–12:20 |
| Kleinberg | COS 423 | MW 11–12:20 |

# 3-dimensional matching

3D-MATCHING. Given $3$ disjoint sets $X$, $Y$, and $Z$, each of size $n$ and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of $n$ triples in $T$ such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

$$X = \{ x_1, x_2, x_3 \}, \quad Y = \{ y_1, y_2, y_3 \}, \quad Z = \{ z_1, z_2, z_3 \}$$

$$T_1 = \{ x_1, y_1, z_2 \}, \quad T_2 = \{ x_1, y_2, z_1 \}, \quad T_3 = \{ x_1, y_2, z_2 \}$$

$$T_4 = \{ x_2, y_2, z_3 \}, \quad T_5 = \{ x_2, y_3, z_3 \},$$

$$T_7 = \{ x_3, y_1, z_3 \}, \quad T_8 = \{ x_3, y_1, z_1 \}, \quad T_9 = \{ x_3, y_2, z_1 \}$$

**an instance of 3d-matching (with n = 3)**

Remark. Generalization of bipartite matching.

# 3-dimensional matching

3D-MATCHING. Given $3$ disjoint sets $X$, $Y$, and $Z$, each of size $n$ and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of $n$ triples in $T$ such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Theorem. 3-SAT $\leq_{\mathrm{P}}$ 3D-MATCHING.

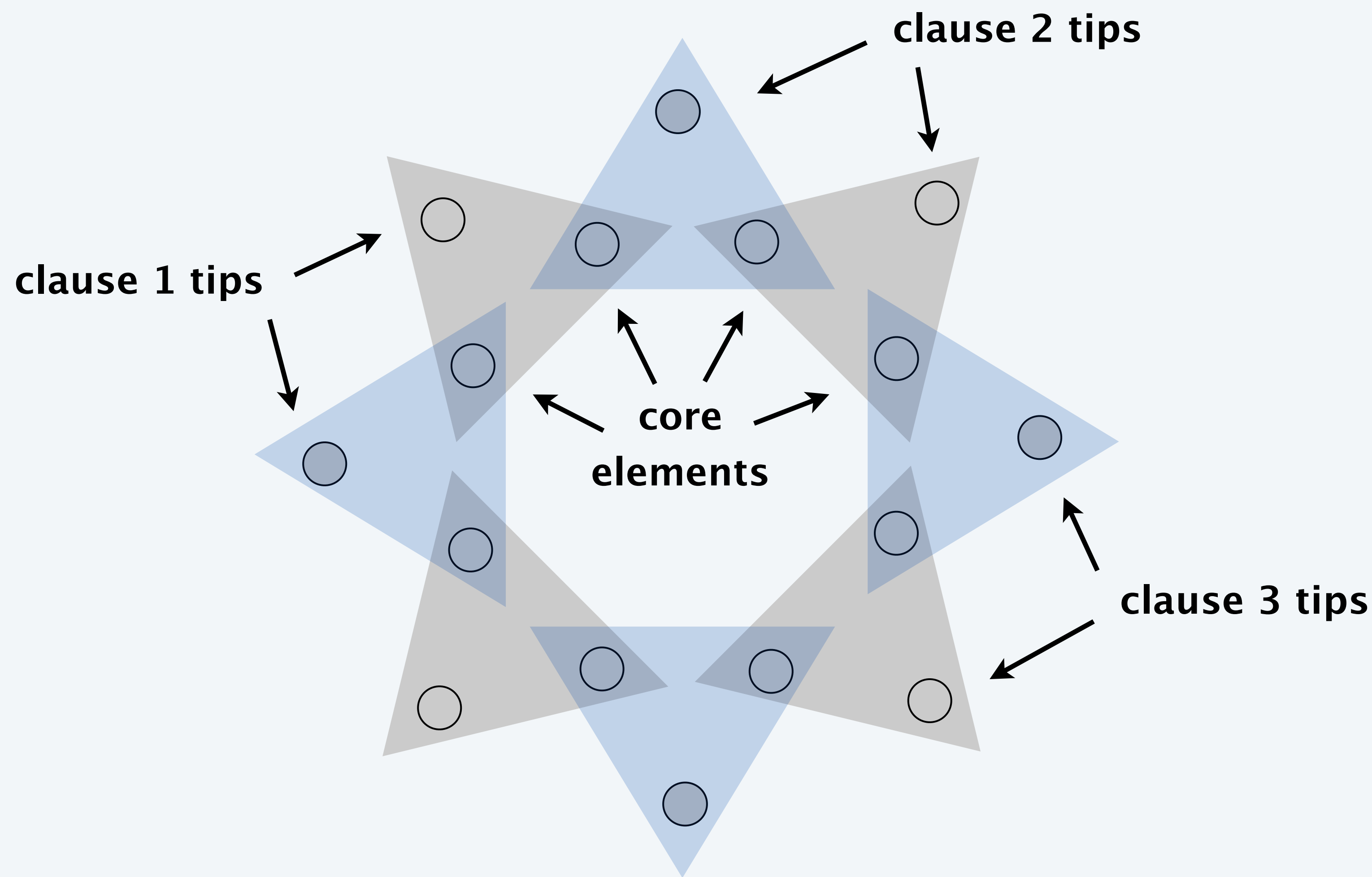
Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance of 3D-MATCHING that has a perfect matching iff $\Phi$ is satisfiable.

# 3-satisfiability reduces to 3-dimensional matching

**Construction.** (part 1)

number of clauses

- Create gadget for each variable $x_i$ with $2k$ core elements and $2k$ tip ones.



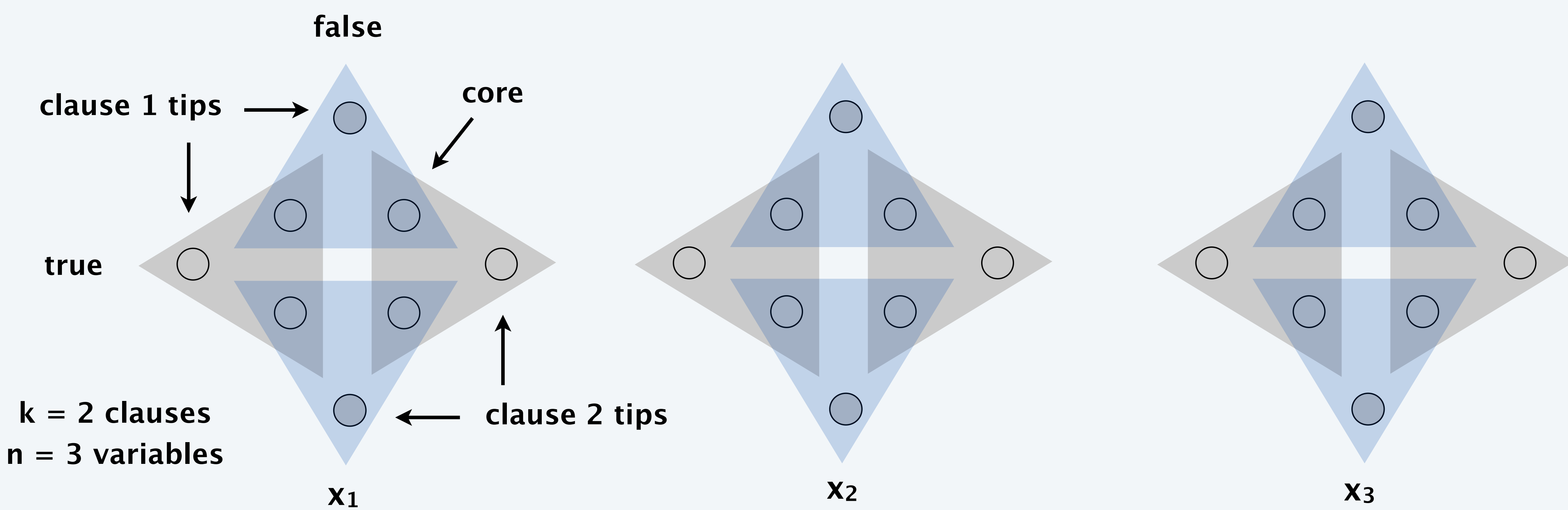

a gadget for variable $x_i$ ($k = 4$)

# 3-satisfiability reduces to 3-dimensional matching

Construction. (part 1)

number of clauses

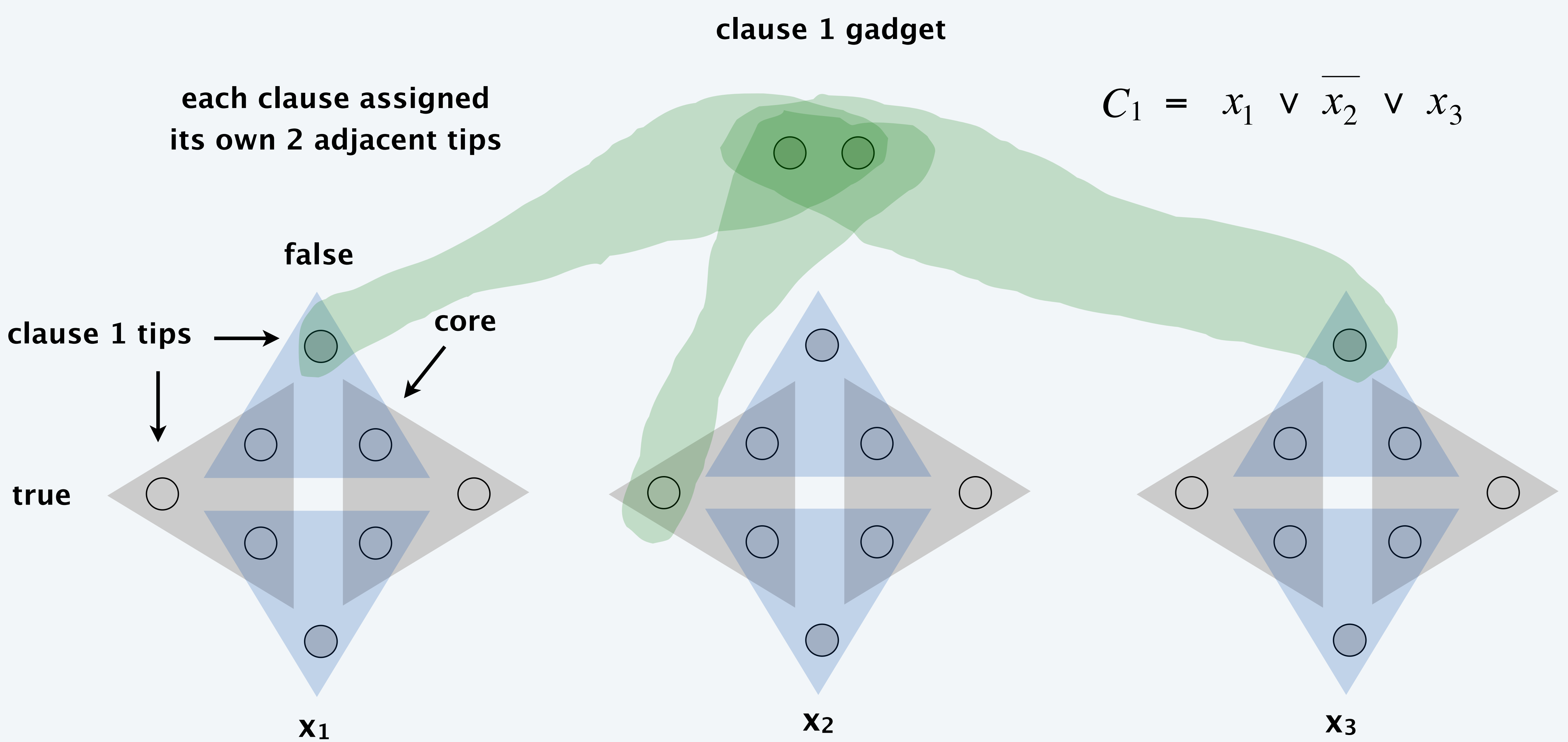
- Create gadget for each variable $x_i$ with $2k$ core elements and $2k$ tip ones.
- No other triples will use core elements.
- In gadget for $x_i$, any perfect matching must use either all gray triples (corresponding to $x_i = true$) or all blue ones (corresponding to $x_i = false$).



# 3-satisfiability reduces to 3-dimensional matching

Construction. (part 2)

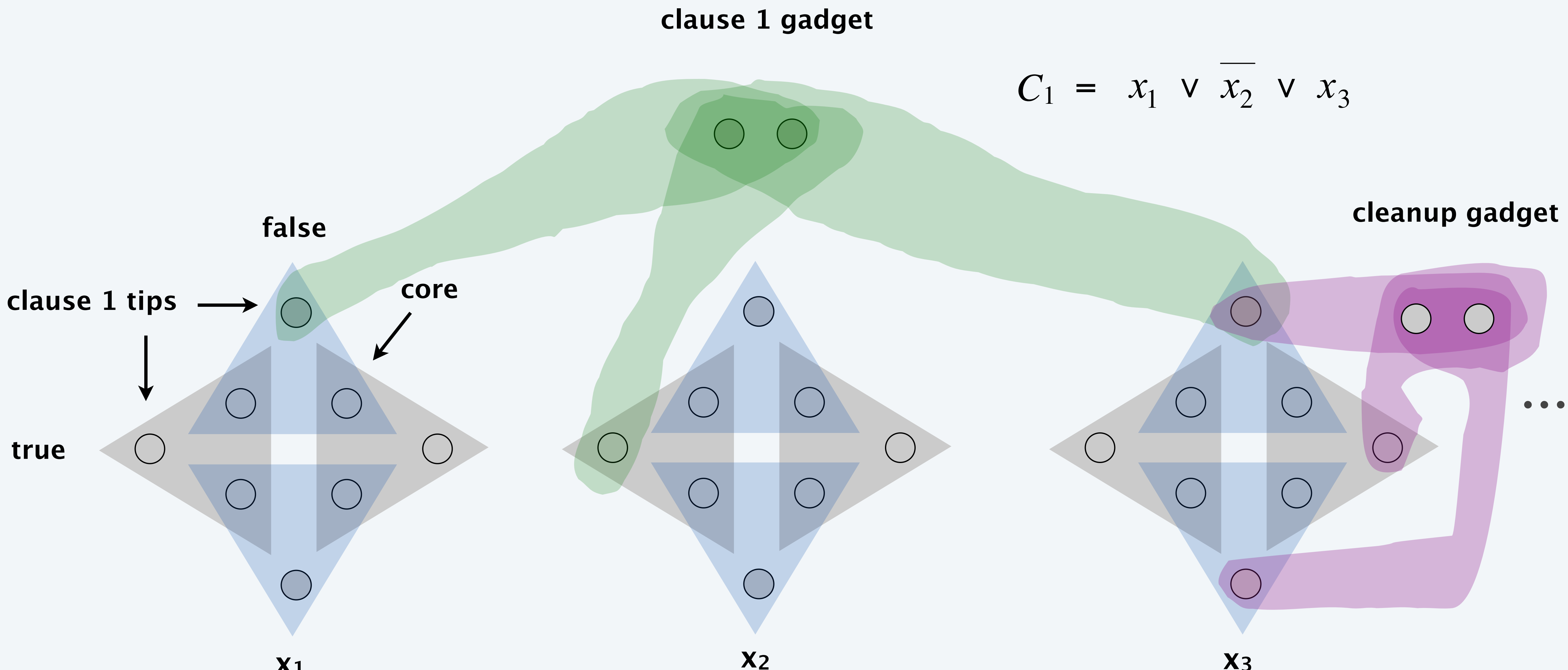
- Create gadget for each clause $C_j$ with two elements and three triples.
- Exactly one of these triples will be used in any 3d-matching.
- Ensures any perfect matching uses either (i) grey core of $x_1$ or (ii) blue core of $x_2$ or (iii) grey core of $x_3$.



# 3-satisfiability reduces to 3-dimensional matching

**Construction.** (part 3)

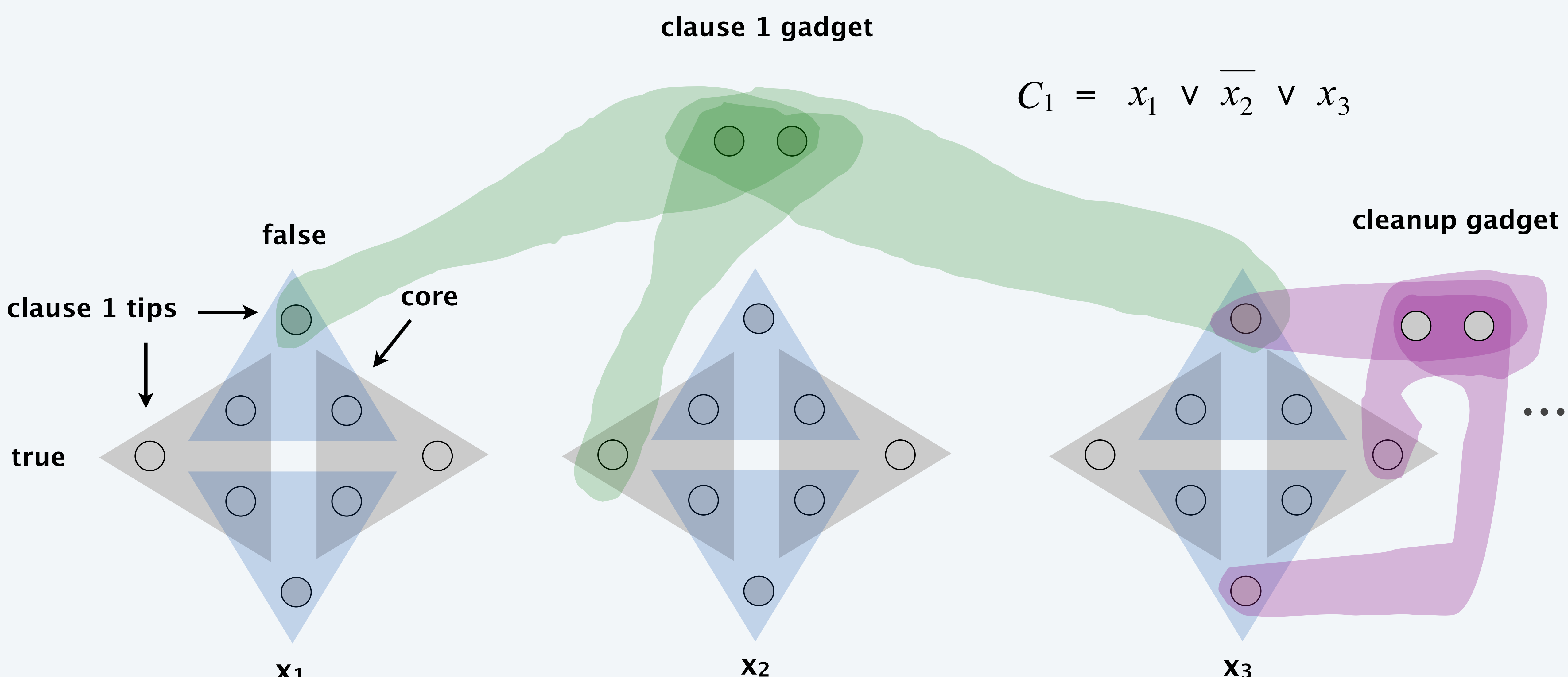
- There are $2nk$ tips: $nk$ covered by blue/gray triples; $k$ by clause triples.
- To cover remaining $(n-1)k$ tips, create $(n-1)k$ cleanup gadgets: same as clause gadget but with $2nk$ triples, connected to every tip.



# 3-satisfiability reduces to 3-dimensional matching

Lemma. Instance $(X, Y, Z)$ has a perfect matching iff $\Phi$ is satisfiable.

Q. What are $X$, $Y$, and $Z$ ?

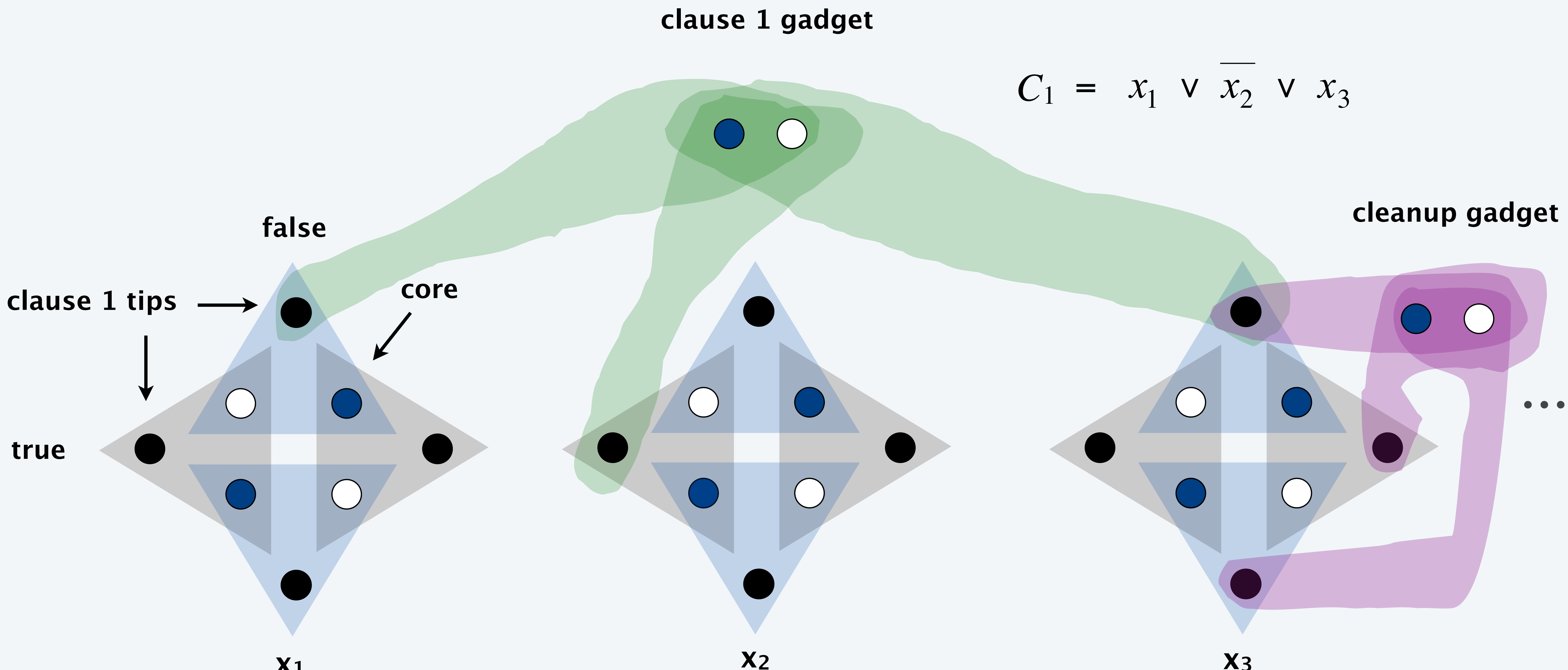


# 3-satisfiability reduces to 3-dimensional matching

Lemma. Instance $(X, Y, Z)$ has a perfect matching iff $\Phi$ is satisfiable.

Q. What are $X$, $Y$, and $Z$ ?

A. $X = black$, $Y = white$, and $Z = blue$.

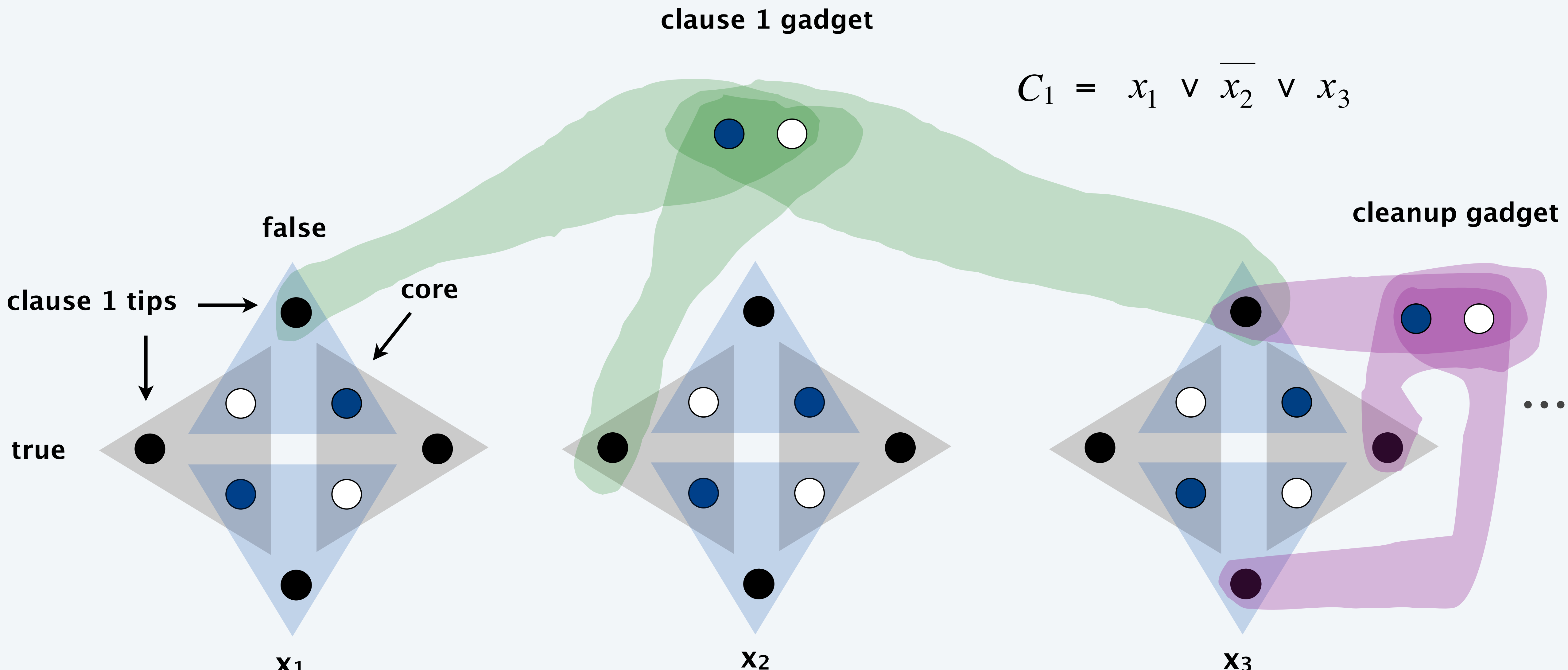


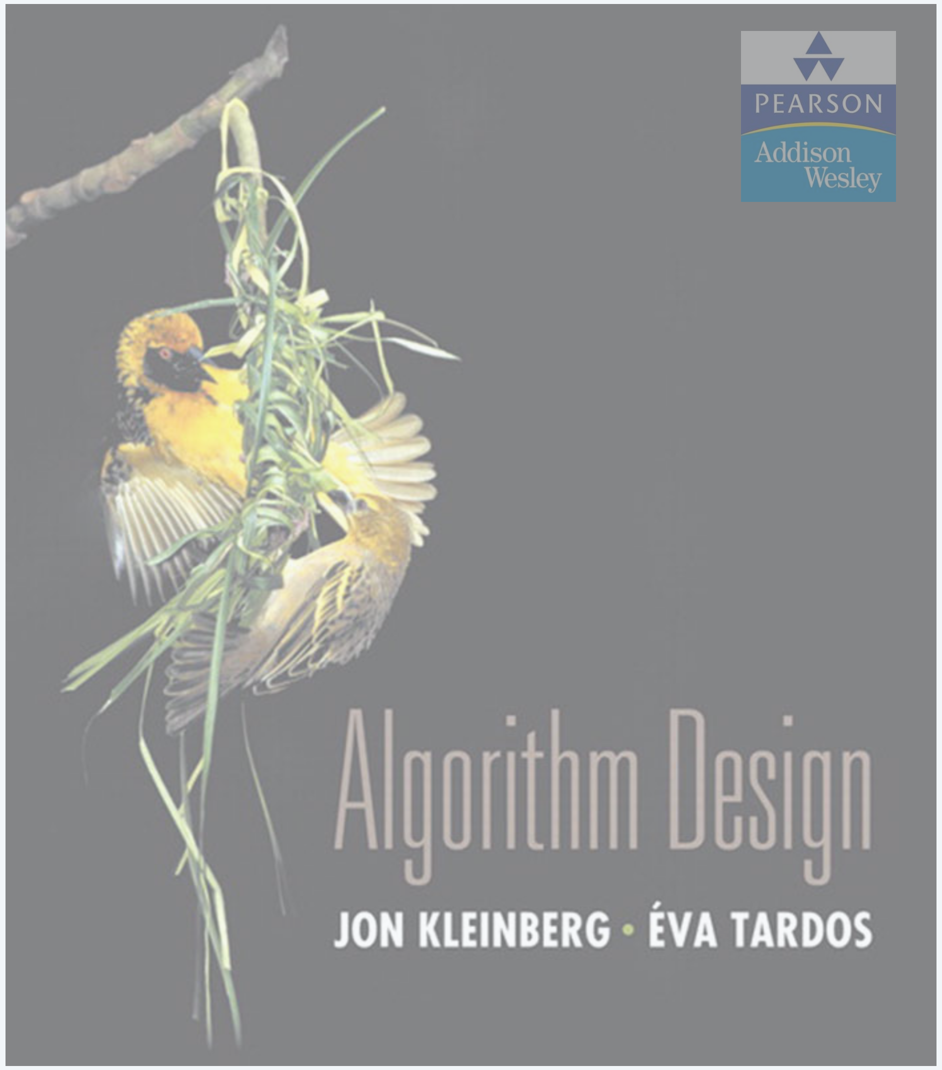
# 3-satisfiability reduces to 3-dimensional matching

**Lemma.** Instance $(X, Y, Z)$ has a perfect matching iff $\Phi$ is satisfiable.

**Pf.** $\Rightarrow$ If 3d-matching, then assign $x_i$ according to gadget $x_i$.

**Pf.** $\Leftarrow$ If $\Phi$ is satisfiable, use any true literal in $C_j$ to select gadget $C_j$ triple. ▪





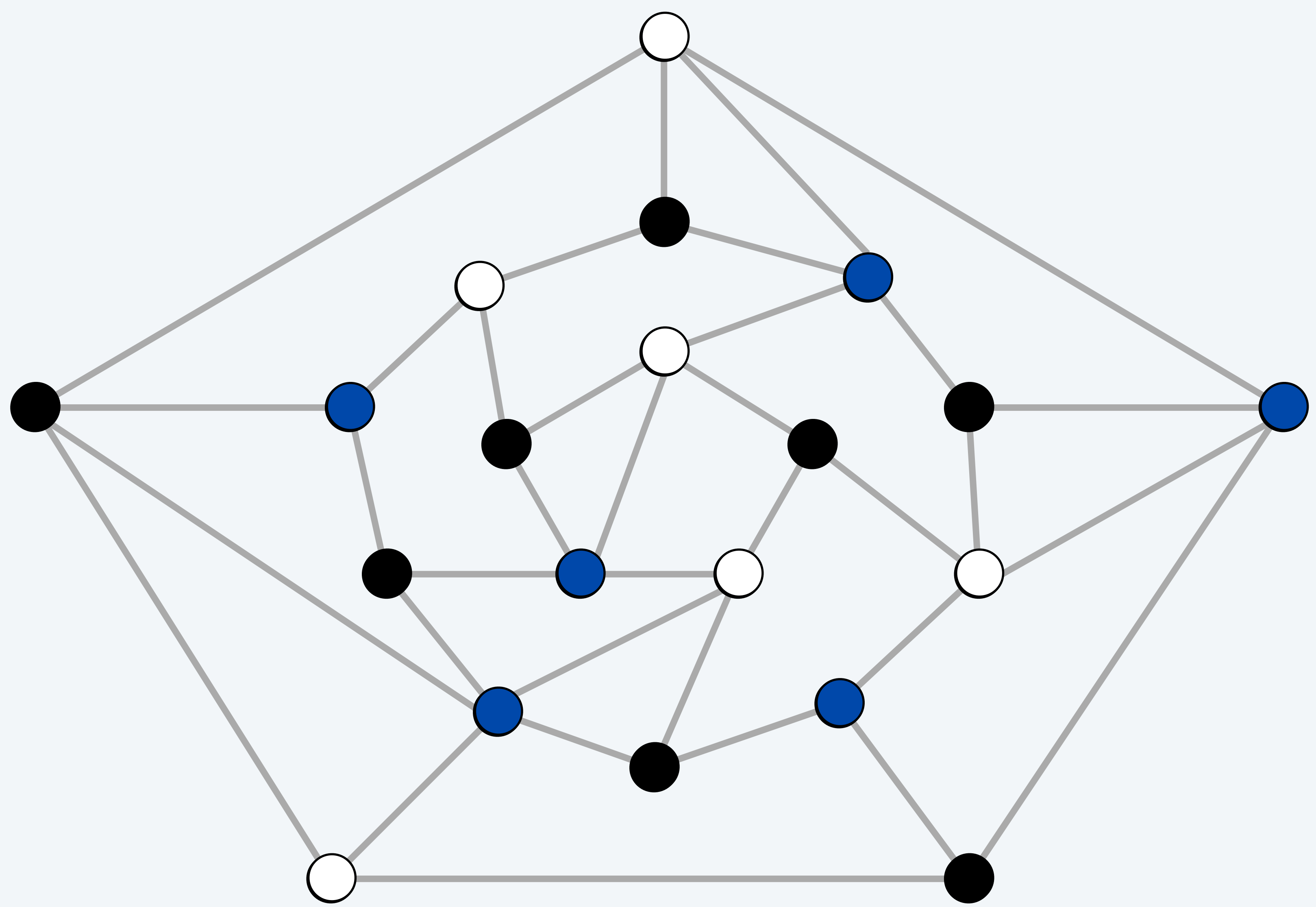

SECTION 8.7

# 8. INTRACTABILITY I

- poly-time reductions
- packing and covering problems
- constraint satisfaction problems
- sequencing problems
- partitioning problems
- *graph coloring*
- numerical problems

# 3-colorability

3-COLOR. Given an undirected graph $G$, can the nodes be colored black, white, and blue so that no adjacent nodes have the same color?



**yes instance**

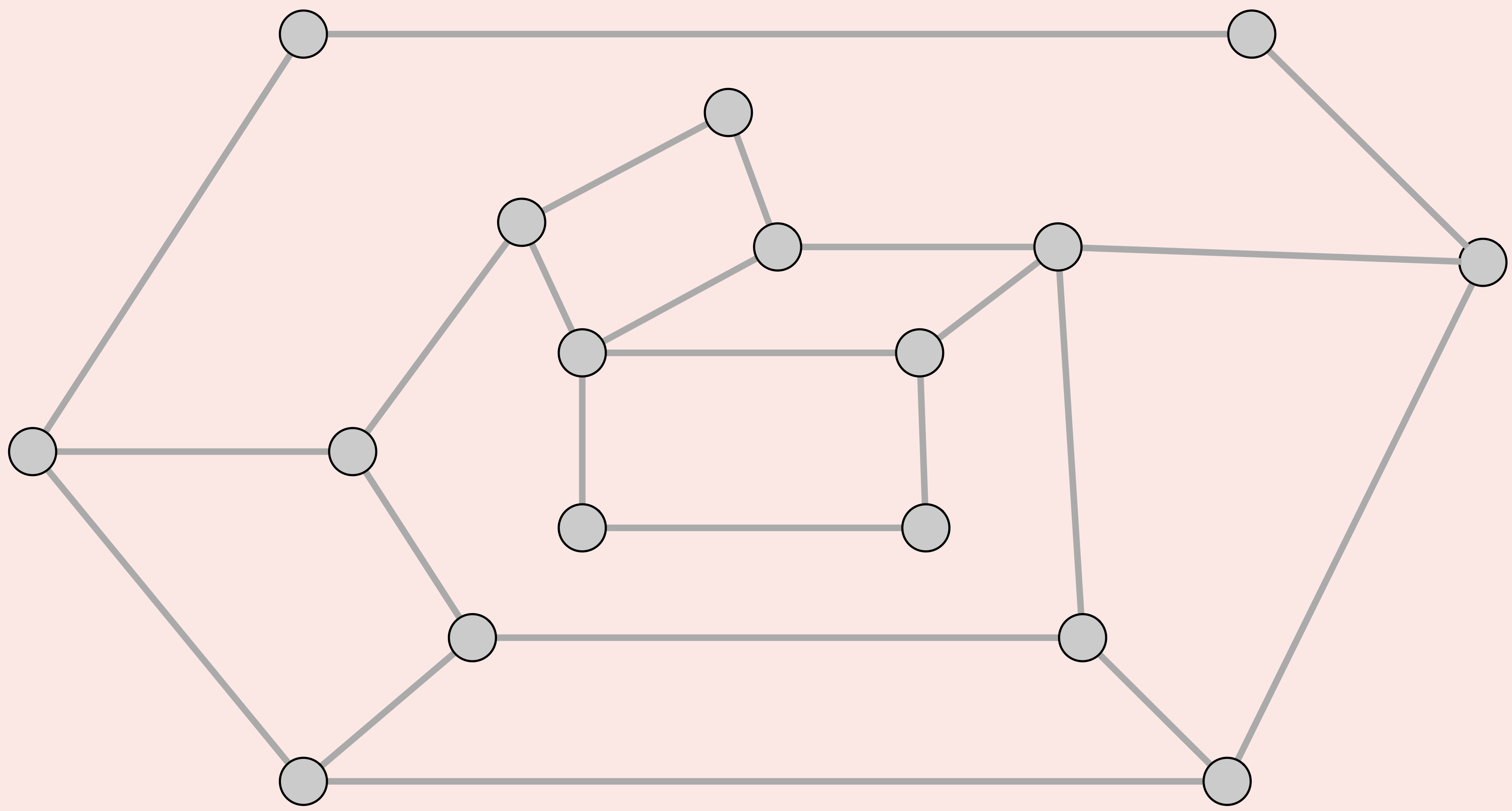
## How difficult to solve 2-COLOR?

**A.** $O(m + n)$ using BFS or DFS.

**B.** $O(mn)$ using maximum flow.

**C.** $\Omega(2^n)$ using brute force.

**D.** Not even Tarjan knows.



# Application: register allocation

**Register allocation.** Assign program variables to machine registers so that no more than $k$ registers are used and no two program variables that are needed at the same time are assigned to the same register.

**Interference graph.** Nodes are program variables; edge between $u$ and $v$ if there exists an operation where both $u$ and $v$ are "live" at the same time.

**Observation.** [Chaitin 1982] Can solve register allocation problem iff interference graph is $k$-colorable.

**Fact.** 3-COLOR $\le_P$ K-REGISTER-ALLOCATION for any constant $k \ge 3$.

REGISTER ALLOCATION & SPILLING VIA GRAPH COLORING

G. J. Chaitin
IBM Research
P.O.Box 218, Yorktown Heights, NY 10598

# 3-satisfiability reduces to 3-colorability

**Theorem.** 3-SAT $\le_P$ 3-COLOR.

**Pf.** Given 3-SAT instance $\Phi$, we construct an instance of 3-COLOR that is 3-colorable iff $\Phi$ is satisfiable.

# 3-satisfiability reduces to 3-colorability

Construction.

(i) Create a graph $G$ with a node for each literal.

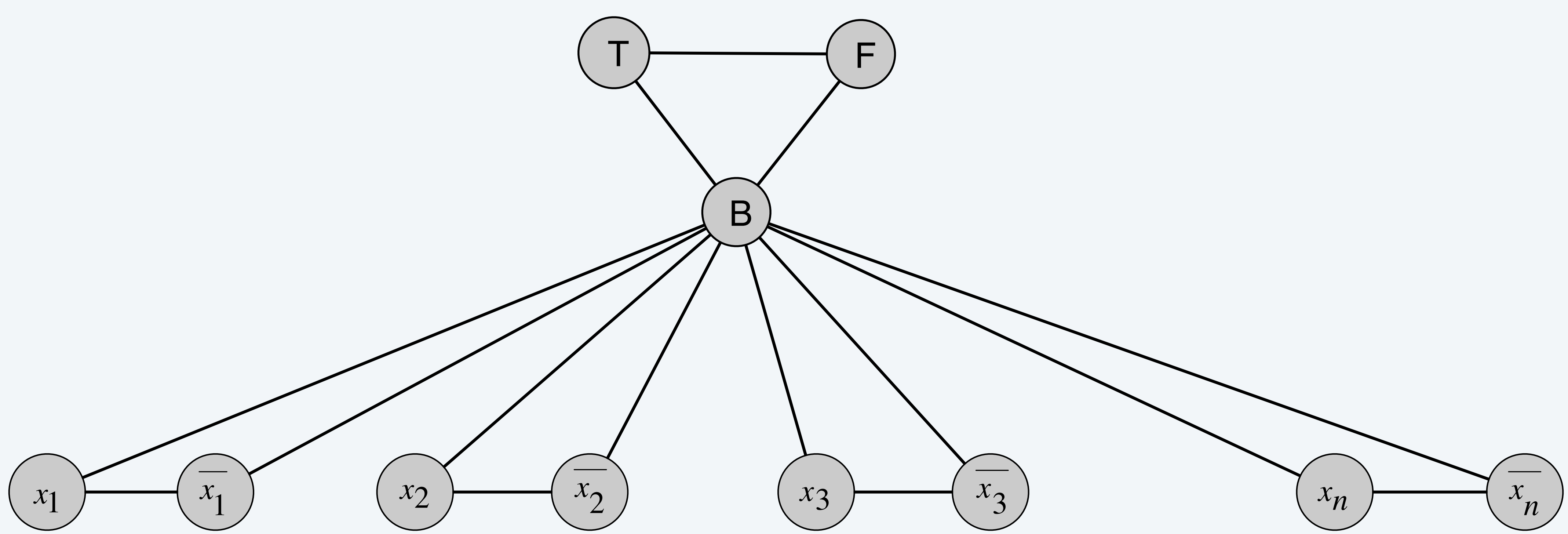
(ii) Connect each literal to its negation.

(iii) Create 3 new nodes $T$, $F$, and $B$; connect them in a triangle.

(iv) Connect each literal to $B$.

(v) For each clause $C_j$, add a gadget of 6 nodes and 13 edges.

to be described later

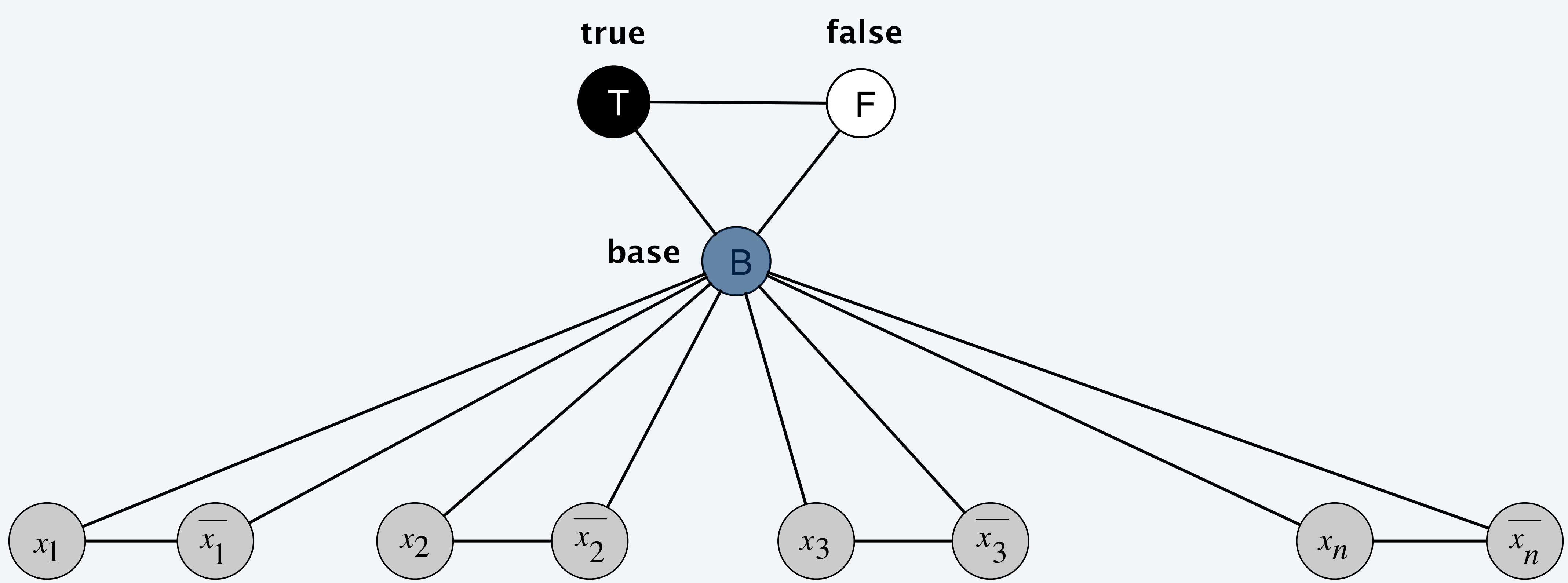


# 3-satisfiability reduces to 3-colorability

Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- WLOG, assume that node $T$ is colored *black*, $F$ is *white*, and $B$ is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).

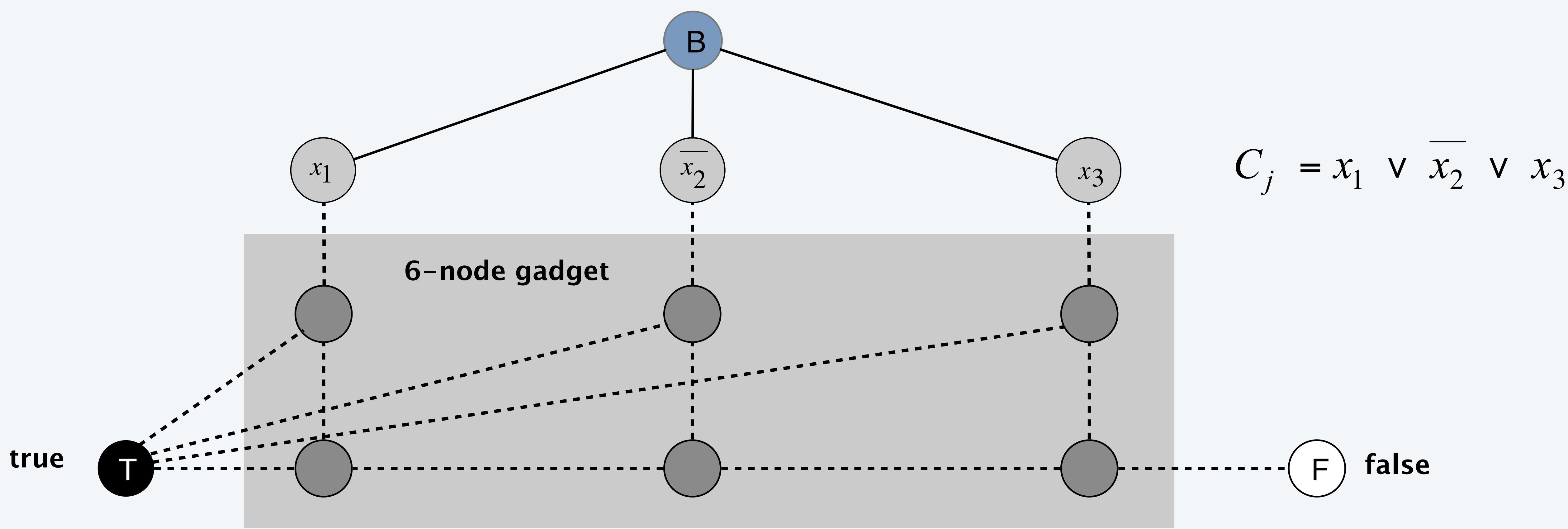


# 3-satisfiability reduces to 3-colorability

Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- WLOG, assume that node $T$ is colored *black*, $F$ is *white*, and $B$ is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).
- (v) ensures at least one literal in each clause is *black*.

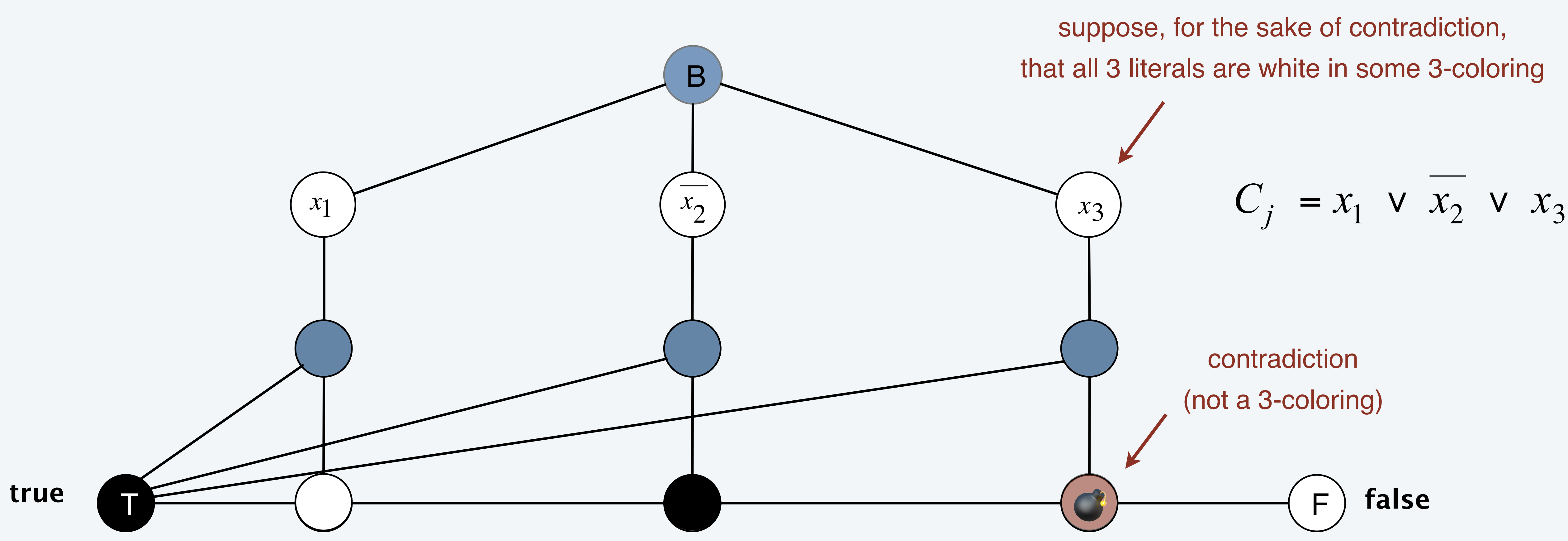


# 3-satisfiability reduces to 3-colorability

Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- WLOG, assume that node $T$ is colored *black*, $F$ is *white*, and $B$ is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).
- (v) ensures at least one literal in each clause is *black*. ▪

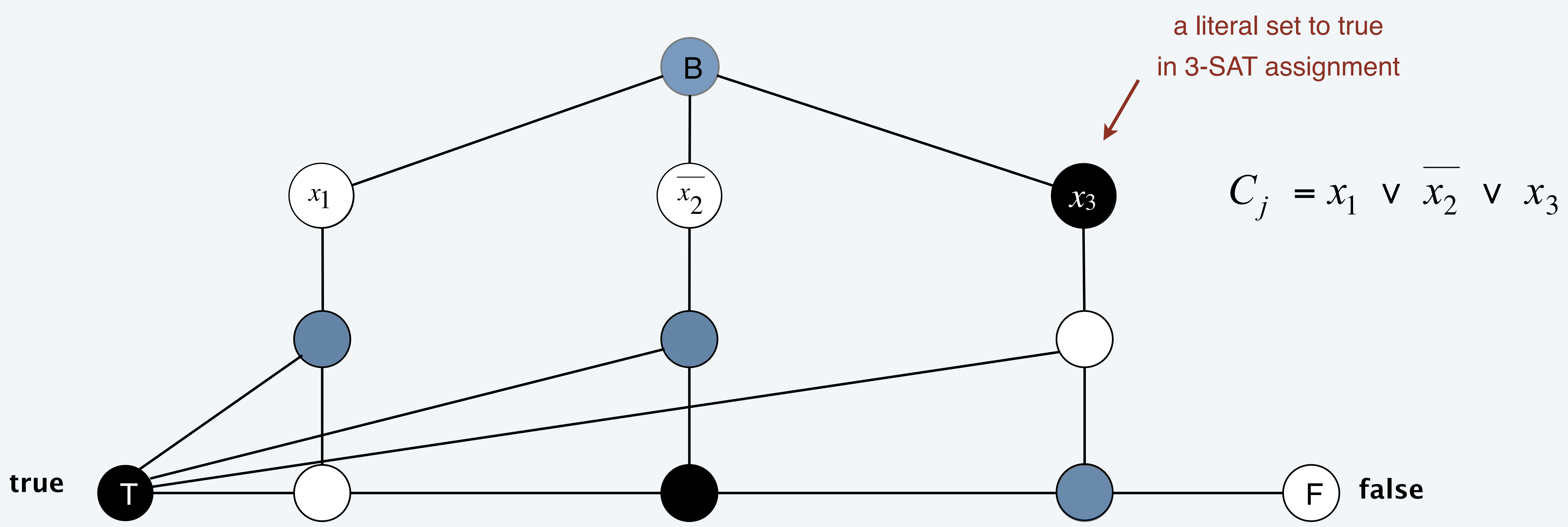


# 3-satisfiability reduces to 3-colorability

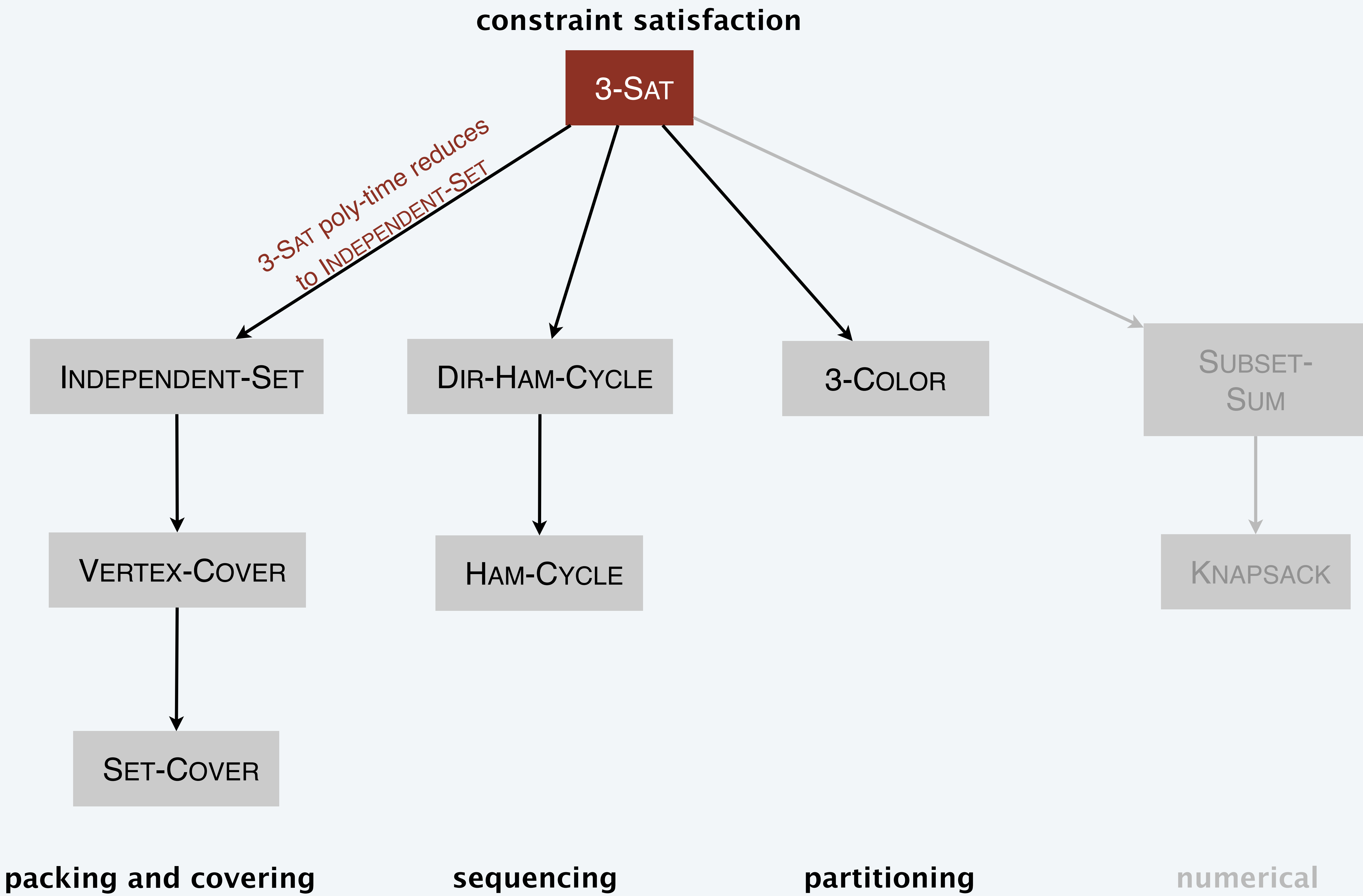
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

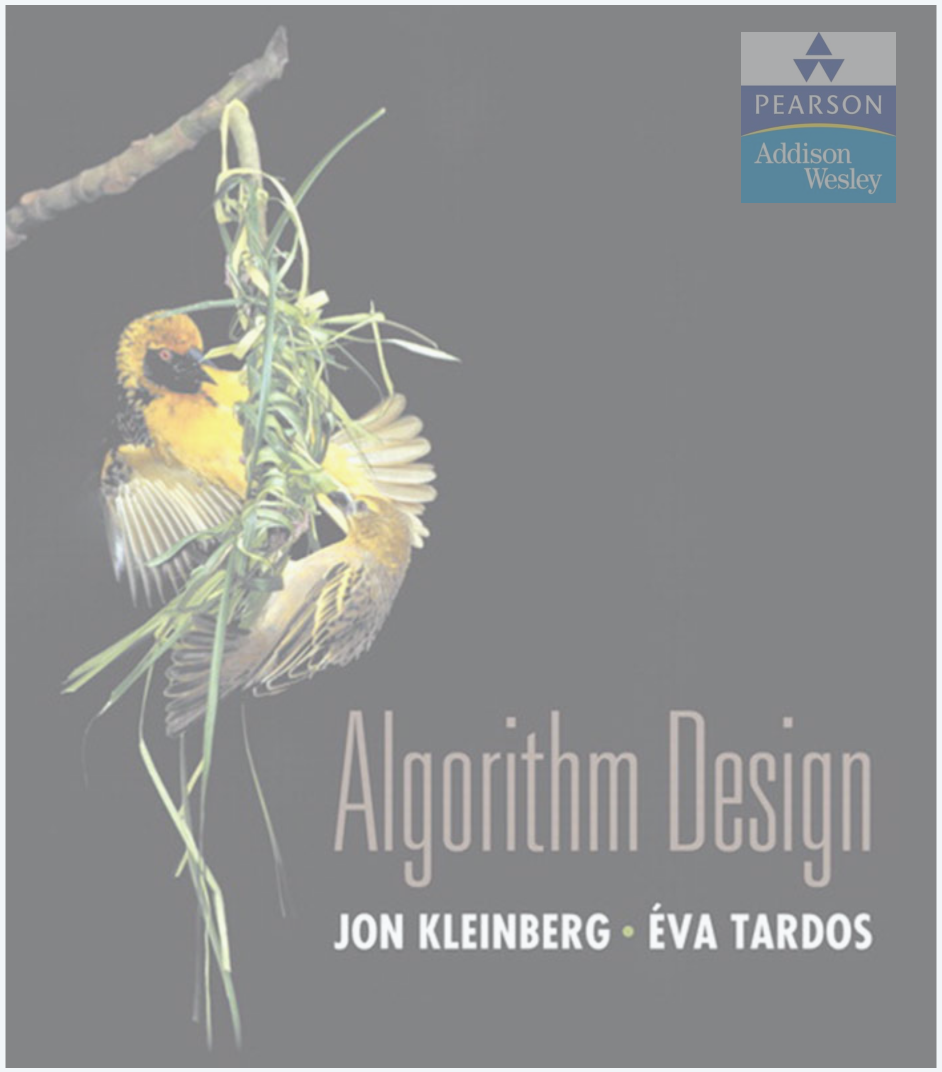
Pf. $\Leftarrow$ Suppose 3-SAT instance $\Phi$ is satisfiable.

- Color all *true* literals *black* and all *false* literals *white*.
- Pick one *true* literal; color node below that node *white*, and node below that *blue*.
- Color remaining middle row nodes *blue*.
- Color remaining bottom nodes *black* or *white*, as forced. ▪



# Poly-time reductions



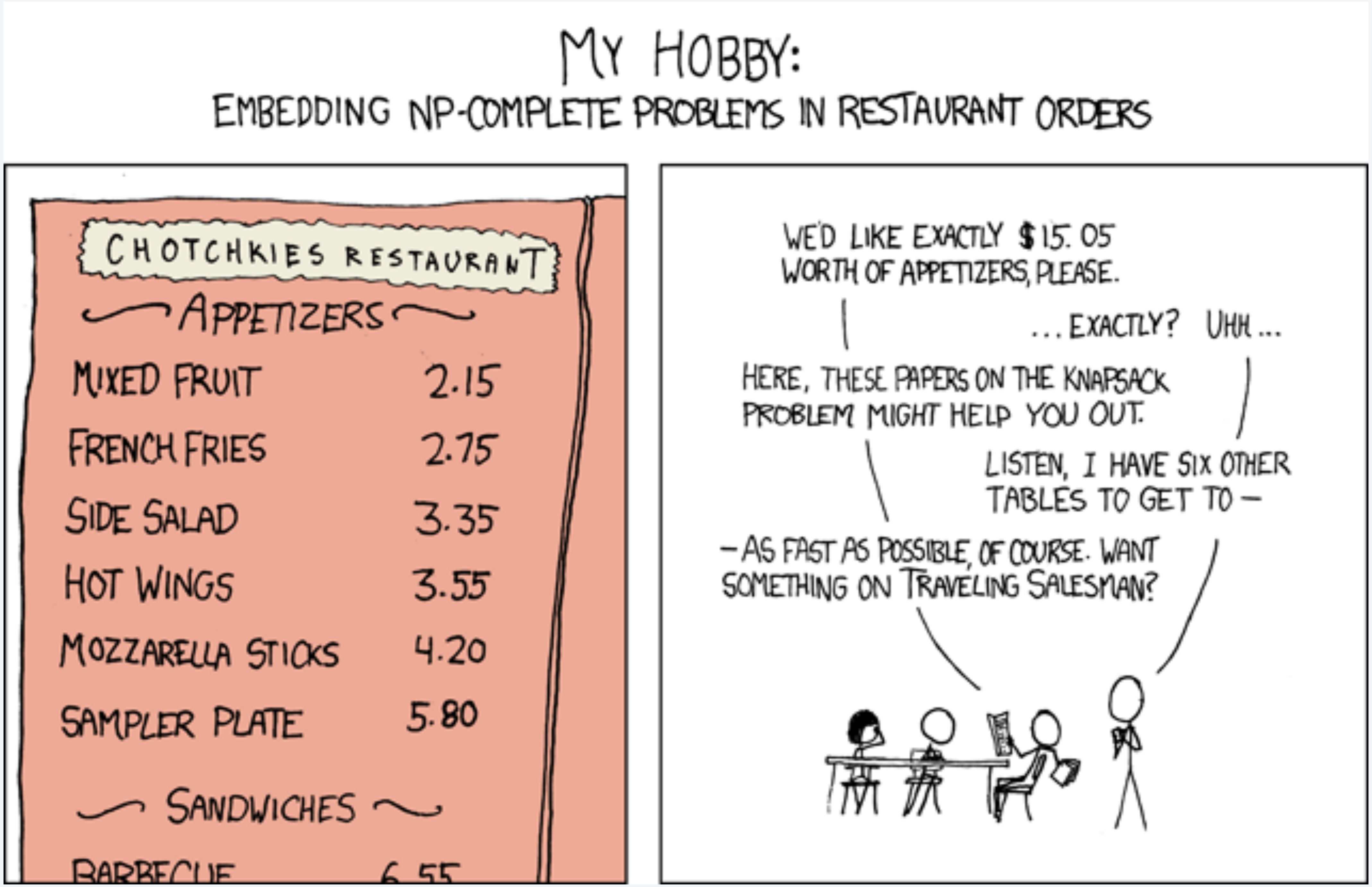



SECTION 8.8

# 8. INTRACTABILITY I

- poly-time reductions
- packing and covering problems
- constraint satisfaction problems
- sequencing problems
- partitioning problems
- graph coloring
- *numerical problems*

# My hobby




NP-Complete by Randall Munro
http://xkcd.com/287

# Subset sum

SUBSET-SUM. Given $n$ natural numbers $w_1, \ldots, w_n$ and an integer $W$, is there a subset that adds up to exactly $W$ ?

Ex. $\{ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 \}, \;\; W = 1505.$

Yes. $215 + 355 + 355 + 580 = 1505.$

Remark. With arithmetic problems, input integers are encoded in binary. Poly-time reduction must be polynomial in binary encoding.

# Subset sum

Theorem. 3-SAT $\leq_P$ SUBSET-SUM.

Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance of SUBSET-SUM that has a solution iff $\Phi$ is satisfiable.

# 3-satisfiability reduces to subset sum

Construction. Given 3-SAT instance $\Phi$ with $n$ variables and $k$ clauses, form $2n + 2k$ decimal integers, each having $n + k$ digits:

- Include one digit for each variable $x_i$ and one digit for each clause $C_j$.
- Include two numbers for each variable $x_i$.
- Include two numbers for each clause $C_j$.
- Sum of each $x_i$ digit is $1$;
  sum of each $C_j$ digit is $4$.

Key property. No carries possible $\Rightarrow$ each digit yields one equation.

$$
\begin{aligned}
C_1 &= \neg x_1 \lor x_2 \lor x_3 \\
C_2 &= x_1 \lor \neg x_2 \lor x_3 \\
C_3 &= \neg x_1 \lor \neg x_2 \lor \neg x_3
\end{aligned}
$$

3-SAT instance

| | $x_1$ | $x_2$ | $x_3$ | $C_1$ | $C_2$ | $C_3$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | 1 | 0 | 100,010 |
| $\neg x_1$ | 1 | 0 | 0 | 1 | 0 | 1 | 100,101 |
| $x_2$ | 0 | 1 | 0 | 1 | 0 | 0 | 10,100 |
| $\neg x_2$ | 0 | 1 | 0 | 0 | 1 | 1 | 10,011 |
| $x_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1,110 |
| $\neg x_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 1,001 |
| dummies to get clause columns to sum to 4 | 0 | 0 | 0 | 1 | 0 | 0 | 100 |
| | 0 | 0 | 0 | 2 | 0 | 0 | 200 |
| | 0 | 0 | 0 | 0 | 1 | 0 | 10 |
| | 0 | 0 | 0 | 0 | 2 | 0 | 20 |
| | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| $W$ | 1 | 1 | 1 | 4 | 4 | 4 | 111,444 |

SUBSET-SUM instance

# 3-satisfiability reduces to subset sum

Lemma. $\Phi$ is satisfiable iff there exists a subset that sums to $W$.

Pf. $\Rightarrow$ Suppose 3-SAT instance $\Phi$ has satisfying assignment $x^*$.

- If $x_i^* = true$, select integer in row $x_i$ ; otherwise, select integer in row $\neg x_i$.
- Each $x_i$ digit sums to $1$.
- Since $\Phi$ is satisfiable, each $C_j$ digit sums to at least $1$ from $x_i$ and $\neg x_i$ rows.
- Select dummy integers to make $C_j$ digits sum to $4$. ▪

$$
\begin{aligned}
C_1 &= \neg x_1 \vee x_2 \vee x_3 \\
C_2 &= x_1 \vee \neg x_2 \vee x_3 \\
C_3 &= \neg x_1 \vee \neg x_2 \vee \neg x_3
\end{aligned}
$$

**3-SAT instance**

| | $x_1$ | $x_2$ | $x_3$ | $C_1$ | $C_2$ | $C_3$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | 1 | 0 | 100,010 |
| $\neg x_1$ | 1 | 0 | 0 | 1 | 0 | 1 | 100,101 |
| $x_2$ | 0 | 1 | 0 | 1 | 0 | 0 | 10,100 |
| $\neg x_2$ | 0 | 1 | 0 | 0 | 1 | 1 | 10,011 |
| $x_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1,110 |
| $\neg x_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 1,001 |
| dummies to get clause columns to sum to 4 | 0 | 0 | 0 | 1 | 0 | 0 | 100 |
| | 0 | 0 | 0 | 2 | 0 | 0 | 200 |
| | 0 | 0 | 0 | 0 | 1 | 0 | 10 |
| | 0 | 0 | 0 | 0 | 2 | 0 | 20 |
| | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| $W$ | 1 | 1 | 1 | 4 | 4 | 4 | 111,444 |

**SUBSET-SUM instance**

# 3-satisfiability reduces to subset sum

Lemma. $\Phi$ is satisfiable iff there exists a subset that sums to $W$.

Pf. $\Leftarrow$ Suppose there exists a subset $S^*$ that sums to $W$.

- Digit $x_i$ forces subset $S^*$ to select either row $x_i$ or row $\neg x_i$ (but not both).
- If row $x_i$ selected, assign $x_i^* = true$ ; otherwise, assign $x_i^* = false$.

Digit $C_j$ forces subset $S^*$ to select at least one literal in clause. ▪

| | $x_1$ | $x_2$ | $x_3$ | $C_1$ | $C_2$ | $C_3$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | 1 | 0 | 100,010 |
| $\neg x_1$ | 1 | 0 | 0 | 1 | 0 | 1 | 100,101 |
| $x_2$ | 0 | 1 | 0 | 1 | 0 | 0 | 10,100 |
| $\neg x_2$ | 0 | 1 | 0 | 0 | 1 | 1 | 10,011 |
| $x_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1,110 |
| $\neg x_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 1,001 |
| dummies to get clause columns to sum to 4 | 0 | 0 | 0 | 1 | 0 | 0 | 100 |
| | 0 | 0 | 0 | 2 | 0 | 0 | 200 |
| | 0 | 0 | 0 | 0 | 1 | 0 | 10 |
| | 0 | 0 | 0 | 0 | 2 | 0 | 20 |
| | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| $W$ | 1 | 1 | 1 | 4 | 4 | 4 | 111,444 |

Subset-Sum instance

$$C_1 = \neg x_1 \lor x_2 \lor x_3$$
$$C_2 = x_1 \lor \neg x_2 \lor x_3$$
$$C_3 = \neg x_1 \lor \neg x_2 \lor \neg x_3$$

3-Sat instance

# SUBSET SUM REDUCES TO KNAPSACK

SUBSET-SUM. Given $n$ natural numbers $w_1, \ldots, w_n$ and an integer $W$, is there a subset that adds up to exactly $W$?

KNAPSACK. Given a set of items $X$, weights $u_i \geq 0$, values $v_i \geq 0$, a weight limit $U$, and a target value $V$, is there a subset $S \subseteq X$ such that:

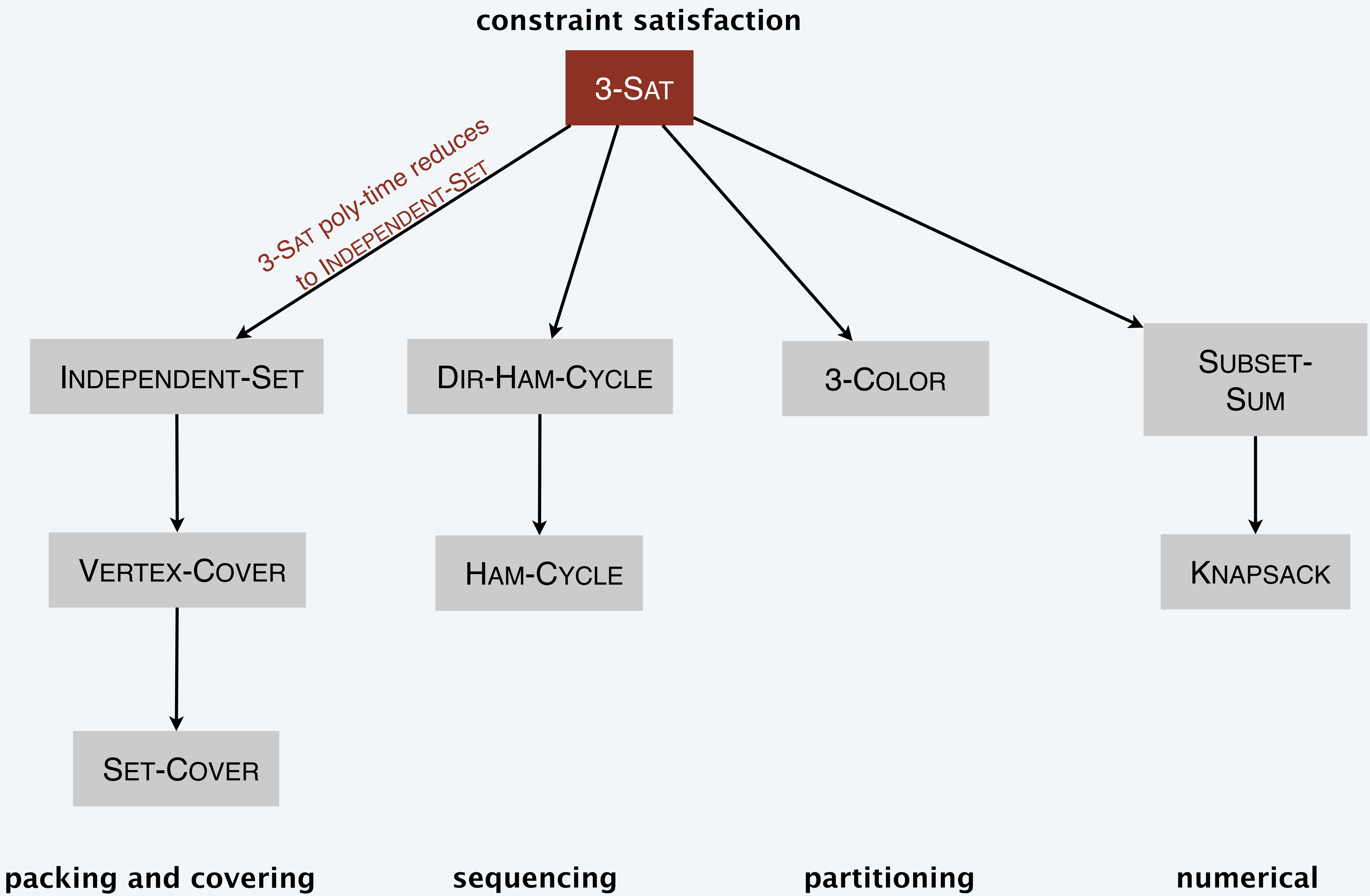
$$\sum_{i \in S} u_i \leq U, \quad \sum_{i \in S} v_i \geq V$$

Recall. $O(n\,U)$ dynamic programming algorithm for KNAPSACK.

Challenge. Prove SUBSET-SUM $\leq_P$ KNAPSACK.

Pf. Given instance $(w_1, \ldots, w_n, W)$ of SUBSET-SUM, create KNAPSACK instance:

# Poly-time reductions



# Karp's 20 poly-time reductions from satisfiability



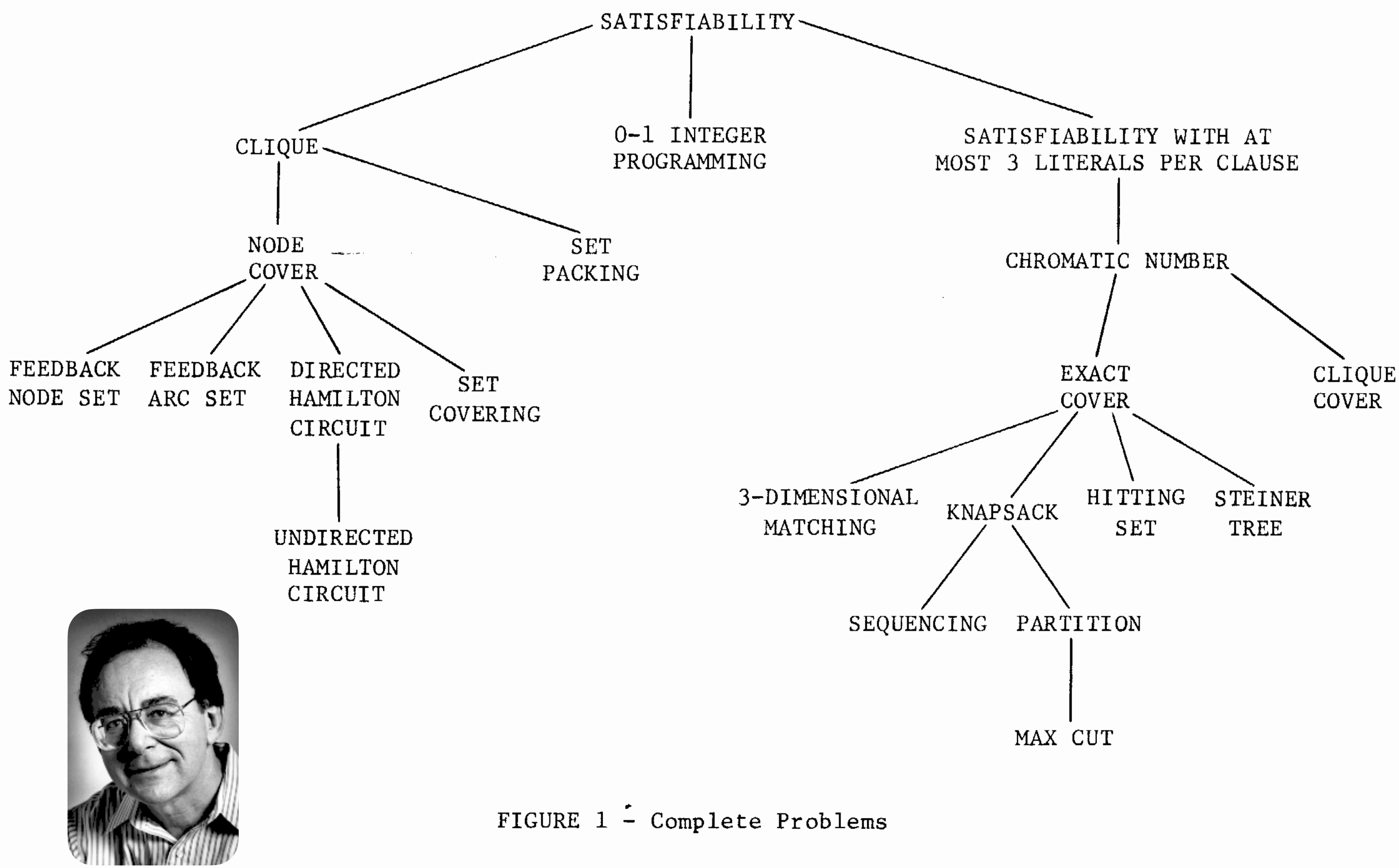



FIGURE 1 - Complete Problems



Dick Karp (1972)
1985 Turing Award