

5. DIVIDE AND CONQUER I

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ *counting inversions*
- ▶ *closest pair of points*

Lecture slides by Kevin Wayne

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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

Divide-and-conquer paradigm

Divide-and-conquer.

- Divide up problem into several subproblems (of the same kind).
- Solve (conquer) each subproblem recursively.
- Combine solutions to subproblems into overall solution.

Most common usage.

- Divide problem of size n into **two** subproblems of size $n/2$. ← $O(n)$ time
- Solve (conquer) two subproblems recursively.
- Combine two solutions into overall solution. ← $O(n)$ time

Consequence.

- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $O(n \log n)$.



attributed to Julius Caesar

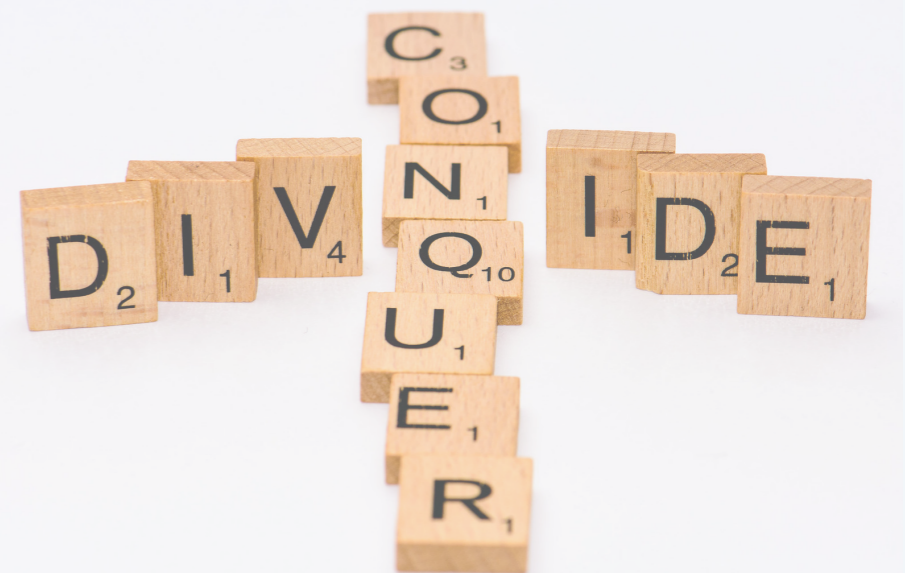
Divide-and-conquer paradigm

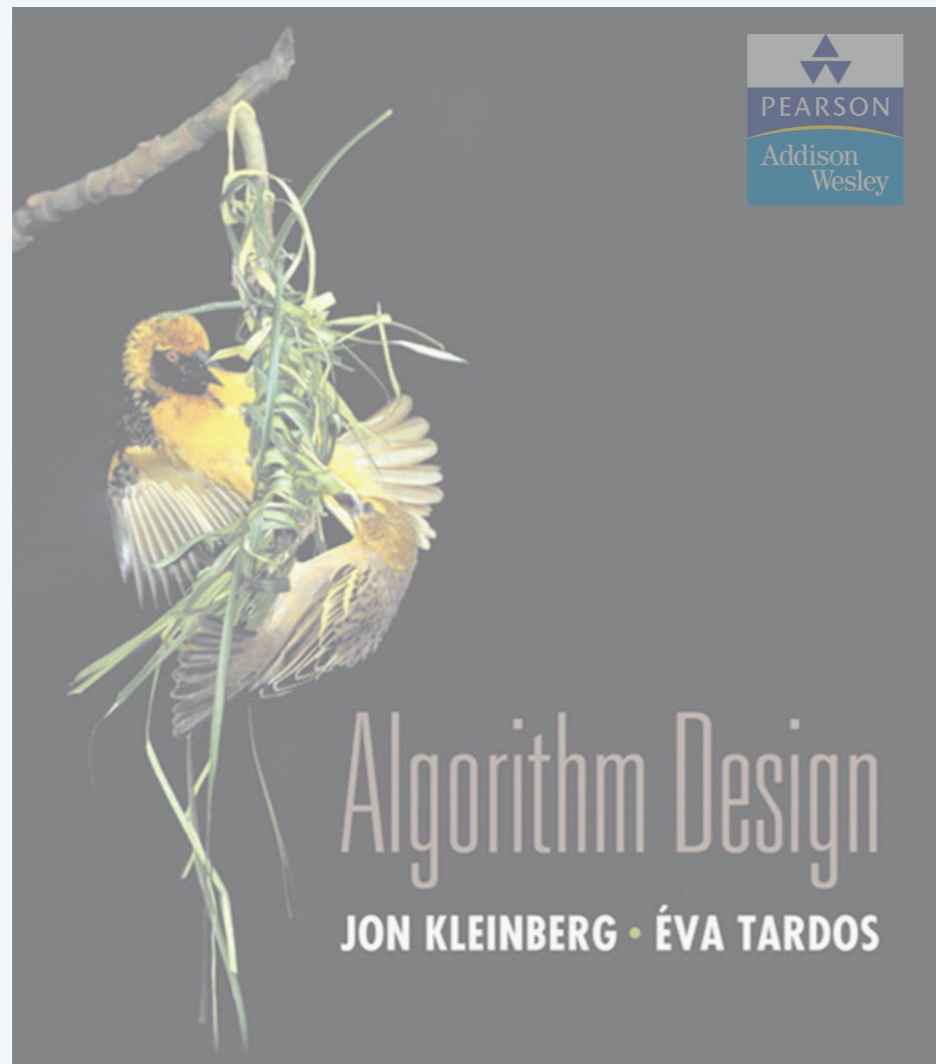
Divide-and-conquer.

- Divide up problem into several subproblems (of the same kind).
- Solve (conquer) each subproblem recursively.
- Combine solutions to subproblems into overall solution.

Examples.

- Mergesort and quicksort (sorting).
- Quickhull and mergehull (convex hull).
- Shamos–Hoey algorithm (closest pair).
- Median-of-medians algorithm (selection).
- Strassen’s algorithm (matrix multiplication).
- Karatsuba’s algorithm (integer multiplication).
- Cooley–Tukey algorithm (fast Fourier transform).
- Ramer–Douglas–Peucker algorithm (decimate a curve).





SECTIONS 5.1–5.2

5. DIVIDE AND CONQUER

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ *counting inversions*
- ▶ *closest pair of points*
- ▶ *integer multiplication*

Sorting problem

Problem. Given a list L of n elements from a totally ordered universe, rearrange them in ascending order.



The screenshot shows a music player interface with a playlist. The playlist is sorted by time, and the song 'Dancing In The Dark' by Bruce Springsteen is highlighted in blue. The interface includes a progress bar at the top and a list of songs below.

	Name	Artist	Time	Album
12	<input checked="" type="checkbox"/> Let It Be	The Beatles	4:03	Let It Be
13	<input checked="" type="checkbox"/> Take My Breath Away	BERLIN	4:13	Top Gun – Soundtrack
14	<input checked="" type="checkbox"/> Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	<input checked="" type="checkbox"/> Dancing With Myself	Billy Idol	4:43	Don't Stop
16	<input checked="" type="checkbox"/> Rebel Yell	Billy Idol	4:49	Rebel Yell
17	<input checked="" type="checkbox"/> Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	<input checked="" type="checkbox"/> Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
19	<input checked="" type="checkbox"/> The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
20	<input checked="" type="checkbox"/> Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	<input checked="" type="checkbox"/> Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	<input checked="" type="checkbox"/> Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	<input checked="" type="checkbox"/> Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	<input checked="" type="checkbox"/> Hurricane	Bob Dylan	8:32	Desire
25	<input checked="" type="checkbox"/> The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	<input checked="" type="checkbox"/> Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	<input checked="" type="checkbox"/> Beds Of Roses	Bon Jovi	6:35	Cross Road
28	<input checked="" type="checkbox"/> Runaway	Bon Jovi	3:53	Cross Road
29	<input checked="" type="checkbox"/> Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	<input checked="" type="checkbox"/> Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	<input checked="" type="checkbox"/> Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	<input checked="" type="checkbox"/> Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	<input checked="" type="checkbox"/> Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	<input checked="" type="checkbox"/> Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	<input checked="" type="checkbox"/> Thunder Road	Bruce Springsteen	4:51	Born To Run
36	<input checked="" type="checkbox"/> Born To Run	Bruce Springsteen	4:30	Born To Run
37	<input checked="" type="checkbox"/> Jungleland	Bruce Springsteen	9:34	Born To Run
38	<input checked="" type="checkbox"/> Turtl Turtl Turtl (To Everything)	The Buds	3:57	Forest Gump The Soundtrack (Disc 2)

Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Scheduling to minimize maximum lateness.
- Minimum spanning trees (Kruskal's algorithm).
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.

input

A	L	G	O	R	I	T	H	M	S
---	---	---	---	---	---	---	---	---	---

sort left half

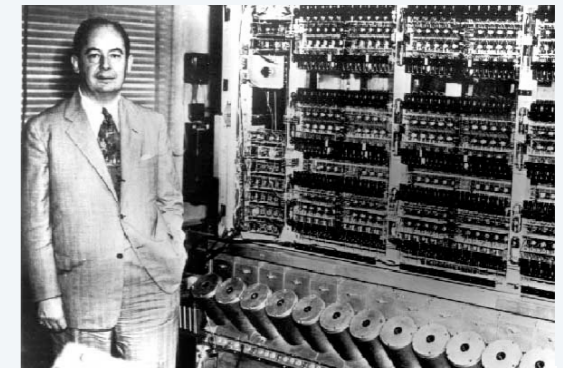
A	G	L	O	R	I	T	H	M	S
---	---	---	---	---	---	---	---	---	---

sort right half

A	G	L	O	R	H	I	M	S	T
---	---	---	---	---	---	---	---	---	---

merge results

A	G	H	I	L	M	O	R	S	T
---	---	---	---	---	---	---	---	---	---



**First Draft
of a
Report on the
EDVAC**

John von Neumann

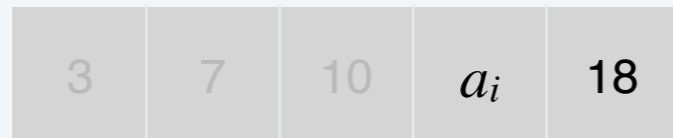
Merging

Goal. Combine two sorted lists A and B into a sorted whole C .

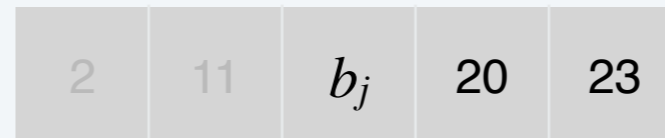


- Scan A and B from left to right.
- Compare a_i and b_j .
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

sorted list A



sorted list B



merge to form sorted list C



Mergesort: implementation

Input. List L of n elements from a totally ordered universe.

Output. The n elements in ascending order.

MERGE-SORT(L)

IF (list L has one element)

RETURN L .

Divide the list into two halves A and B .

$A \leftarrow$ **MERGE-SORT**(A). $\longleftarrow T(n/2)$

$B \leftarrow$ **MERGE-SORT**(B). $\longleftarrow T(n/2)$

$L \leftarrow$ **MERGE**(A, B). $\longleftarrow \Theta(n)$

RETURN L .

Mergesort: proof of correctness

Proposition. Mergesort sorts any list of n elements.

Pf. [by strong induction on n]

- Base case: $n = 1$.
- Inductive hypothesis: assume true for $1, 2, \dots, n-1$.
- By inductive hypothesis, mergesort sorts both left and right halves.
- Merging operation combines two sorted lists into a sorted whole. ■

A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of length n .

Recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n > 1 \end{cases}$$

between $\lfloor n/2 \rfloor$ and $n - 1$ compares

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to solve this recurrence.

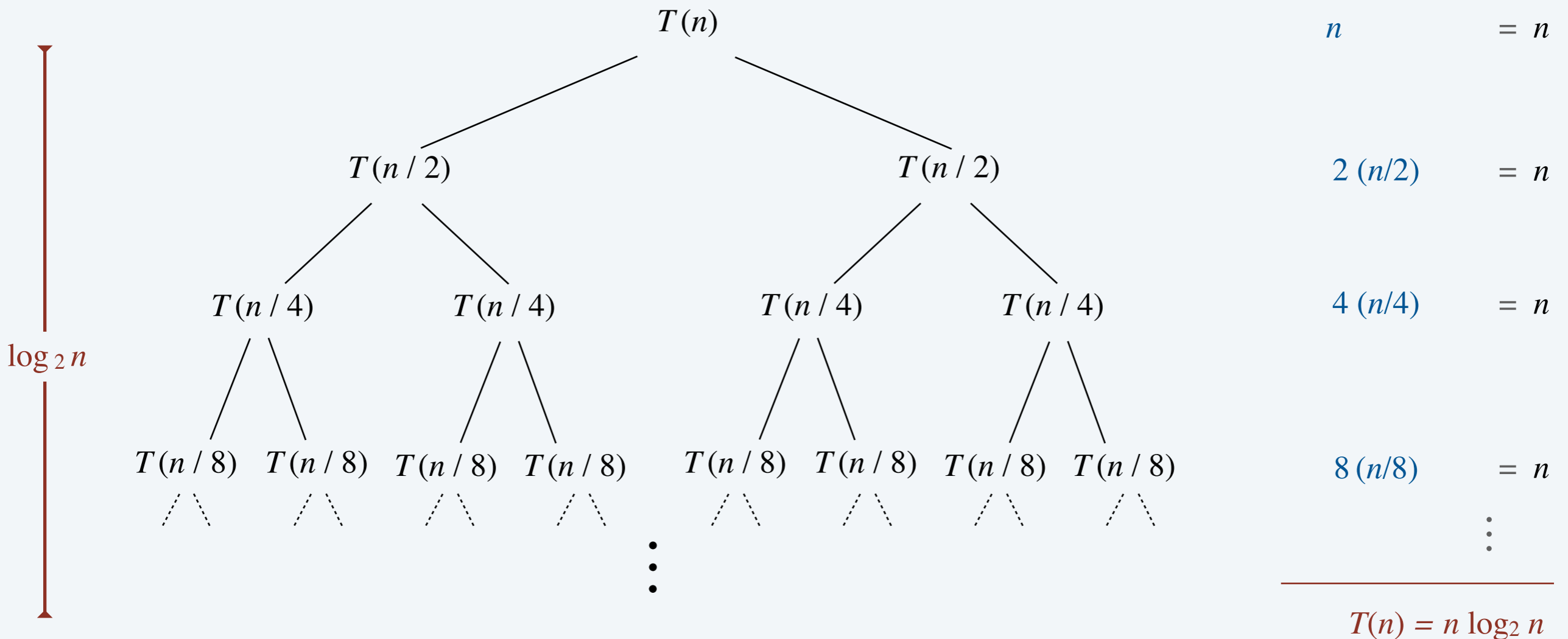
Initially we assume n is a power of 2 and replace \leq with $=$ in the recurrence.

Divide-and-conquer recurrence: recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

assuming n is a power of 2



Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

assuming n
is a power of 2

Pf. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0 = n \log_2 n$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) & \stackrel{\text{recurrence}}{=} 2T(n) + 2n \\ & \stackrel{\text{inductive hypothesis}}{\longrightarrow} = 2n \log_2 n + 2n \\ & = 2n (\log_2 (2n) - 1) + 2n \\ & = 2n \log_2 (2n). \quad \blacksquare \end{aligned}$$

Analysis of mergesort recurrence

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) \leq n \lceil \log_2 n \rceil$.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n > 1 \end{cases}$$

no longer assuming n is a power of 2

Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n / 2 \rfloor$ and $n_2 = \lceil n / 2 \rceil$ and note that $n = n_1 + n_2$.
- Induction step: assume true for $1, 2, \dots, n - 1$.

$$T(n) \leq T(n_1) + T(n_2) + n$$

inductive hypothesis \longrightarrow $\leq n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n$

$$\leq n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n$$

$$= n \lceil \log_2 n_2 \rceil + n$$

$$\leq n (\lceil \log_2 n \rceil - 1) + n$$

$$= n \lceil \log_2 n \rceil. \quad \blacksquare$$

$$n_2 = \lceil n/2 \rceil$$

$$\leq \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil$$

$$= 2^{\lceil \log_2 n \rceil} / 2$$

$$\log_2 n_2 \leq \lceil \log_2 n \rceil - 1$$

an integer

Digression: sorting lower bound

Challenge. How to prove a lower bound for **all** conceivable algorithms?

Model of computation. Comparison trees.

- Can access the elements only through pairwise comparisons.
- All other operations (control, data movement, etc.) are free.

Cost model. Number of compares.

Q. Realistic model?

A1. Yes. Java, Python, C++, ...

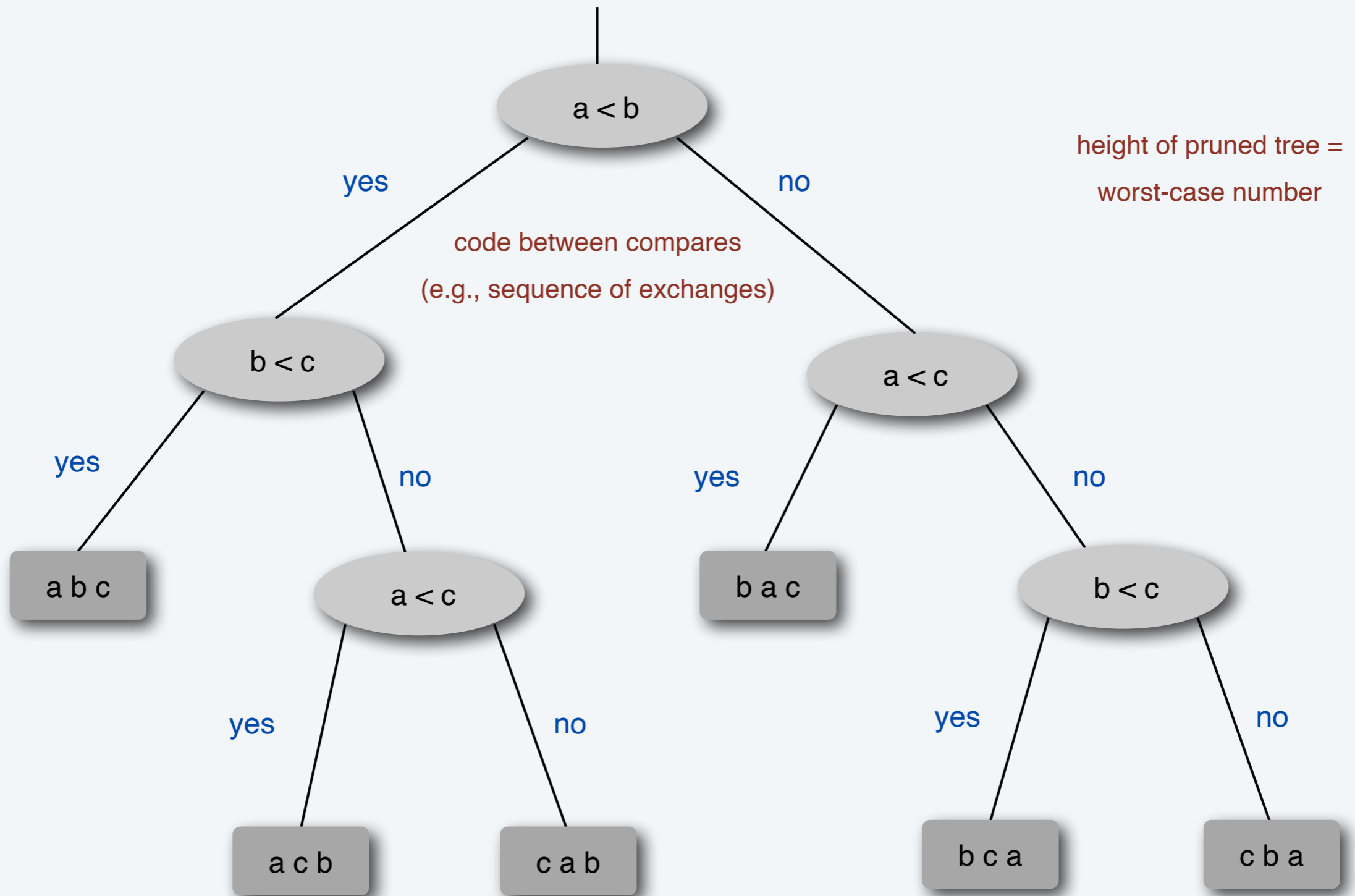
A2. Yes. Mergesort, insertion sort, quicksort, heapsort, ...

A3. No. Bucket sort, radix sorts, ...

`sort(*, key=None, reverse=False)`

This method sorts the list in place, using only `<` comparisons between items. Exceptions are not suppressed – if any comparison operations fail, the entire sort operation will fail (and the list will likely be left in a partially modified state).

Comparison tree (for 3 distinct keys a, b, and c)



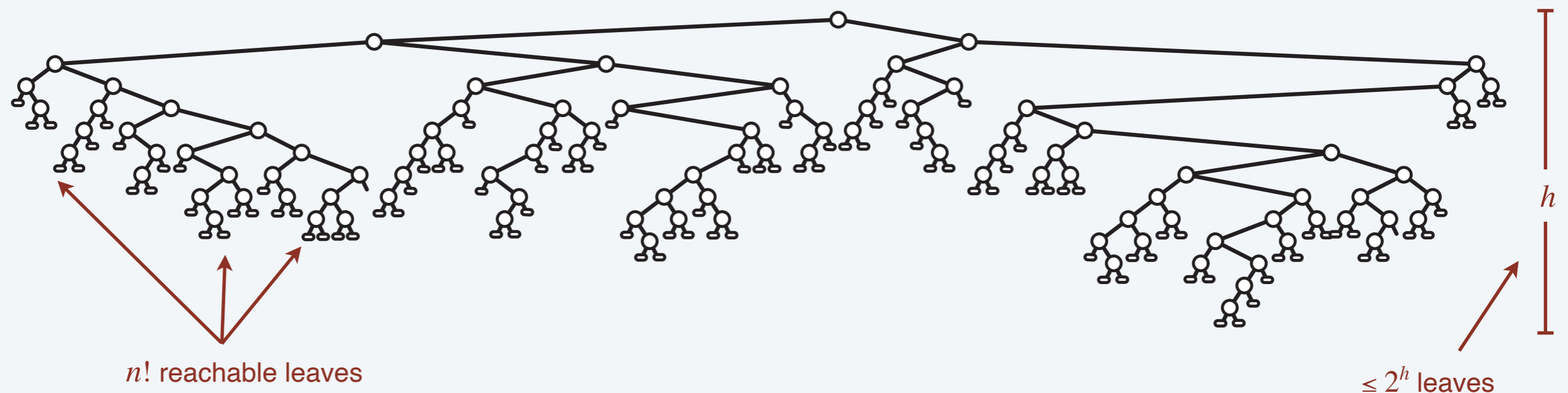
each reachable leaf corresponds to one (and only one) ordering;
exactly one reachable leaf for each possible ordering

Sorting lower bound

Theorem. Any deterministic compare-based sorting algorithm must make $\Omega(n \log n)$ compares in the worst-case.

Pf. [information theoretic]

- Assume array consists of n distinct values a_1 through a_n .
- Worst-case number of compares = height h of pruned comparison tree.
- Binary tree of height h has $\leq 2^h$ leaves.
- $n!$ different orderings $\Rightarrow n!$ reachable leaves.



Sorting lower bound

Theorem. Any deterministic compare-based sorting algorithm must make $\Omega(n \log n)$ compares in the worst-case.

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- Assume array consists of n distinct values a_1 through a_n .
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- Binary tree of height h has $\leq 2^h$ leaves.
- $n!$ different orderings $\Rightarrow n!$ reachable leaves.

$$2^h \geq \# \text{ reachable leaves} = n!$$

$$\Rightarrow h \geq \log_2(n!)$$

$$\geq n \log_2 n - n / \ln 2 \quad \blacksquare$$



Stirling's formula



Note. Lower bound can be extended to include randomized algorithms.

SHUFFLING A LINKED LIST



Problem. Given a singly linked list, rearrange its nodes uniformly at random.

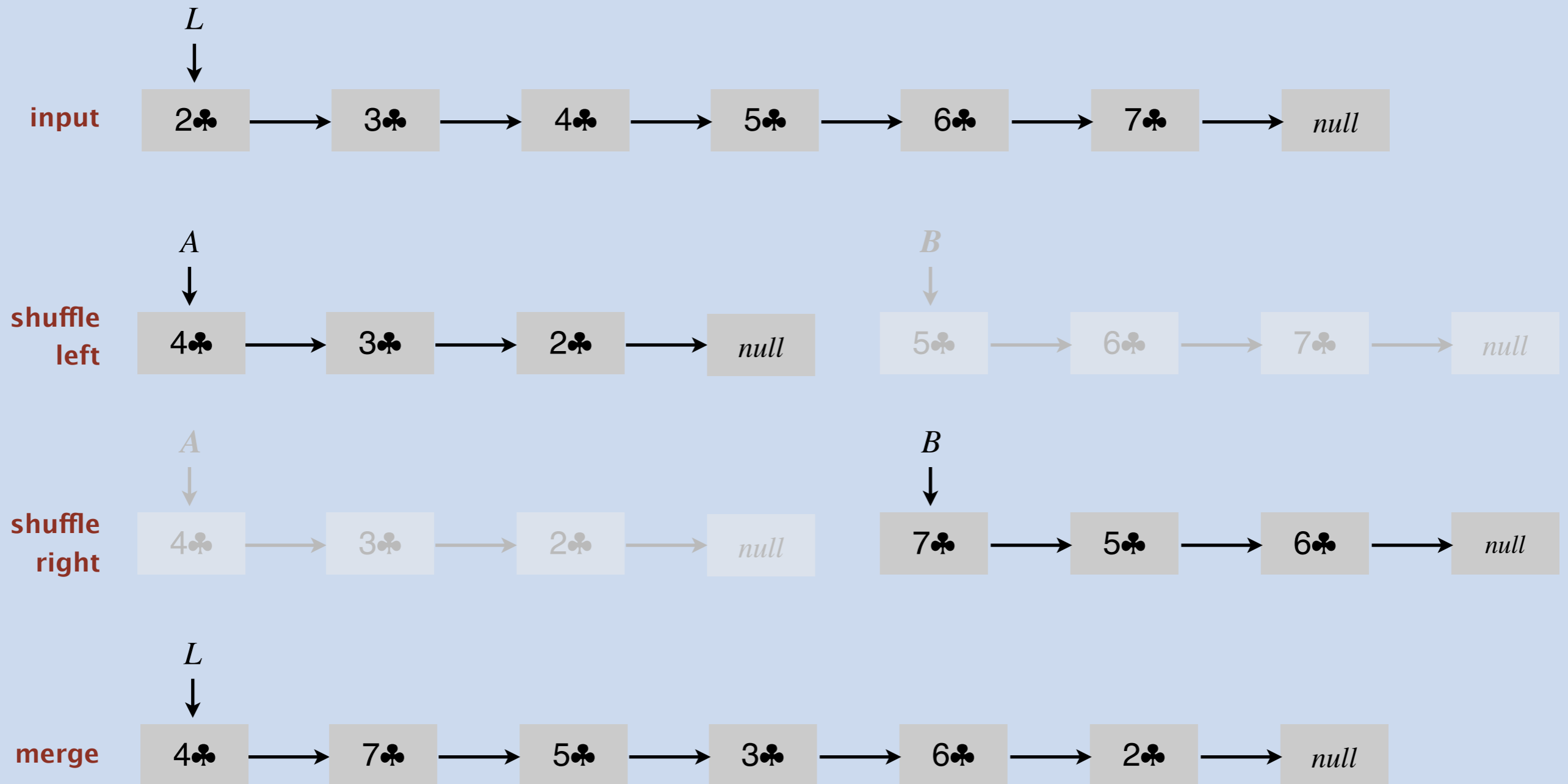
Assumption. Access to a perfect random-number generator.

all $n!$ permutations
equally likely

Performance. $O(n \log n)$ time, $O(\log n)$ extra space.



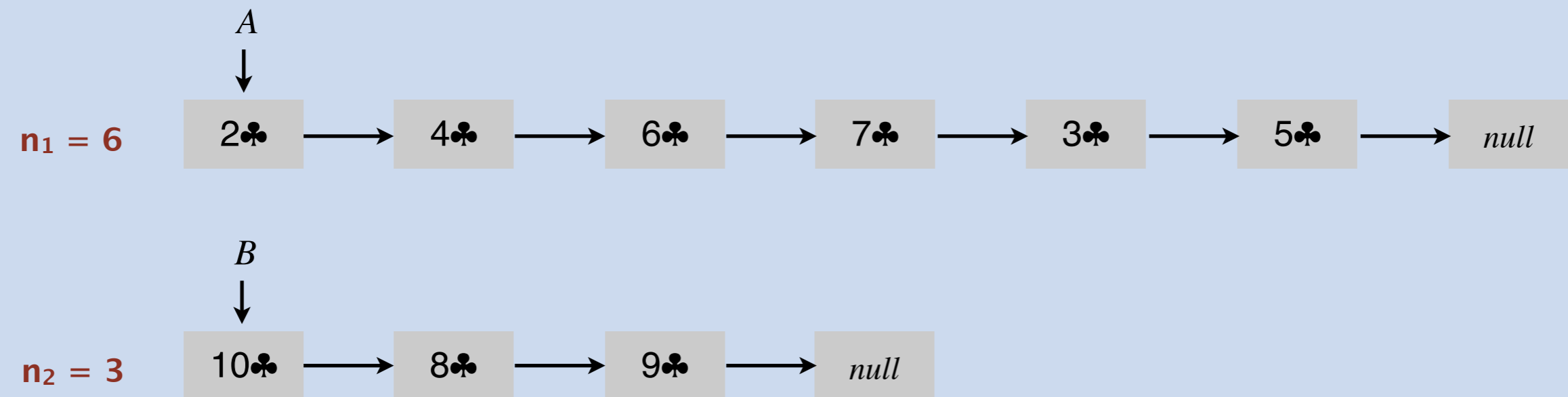
Merge-shuffling a linked list



Proposition. Assuming that the merge can be done in $O(n)$ time, algorithm shuffles a linked list in $O(n \log n)$ time using $O(\log n)$ extra space.

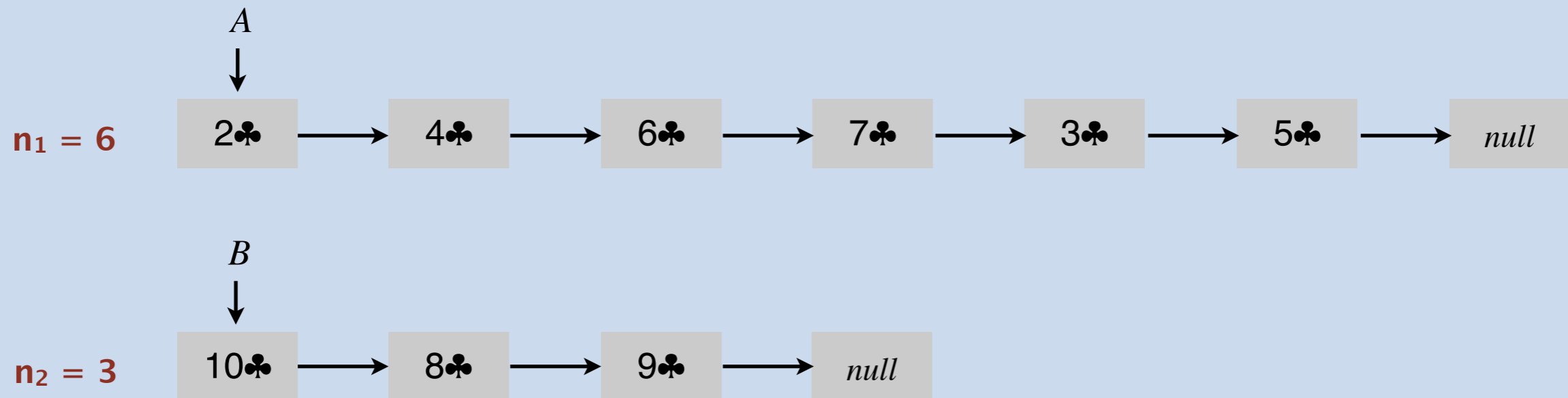
Merging two shuffled linked lists

Problem. Given two shuffled linked lists, create a shuffled linked list whole.

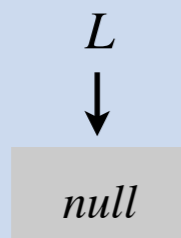


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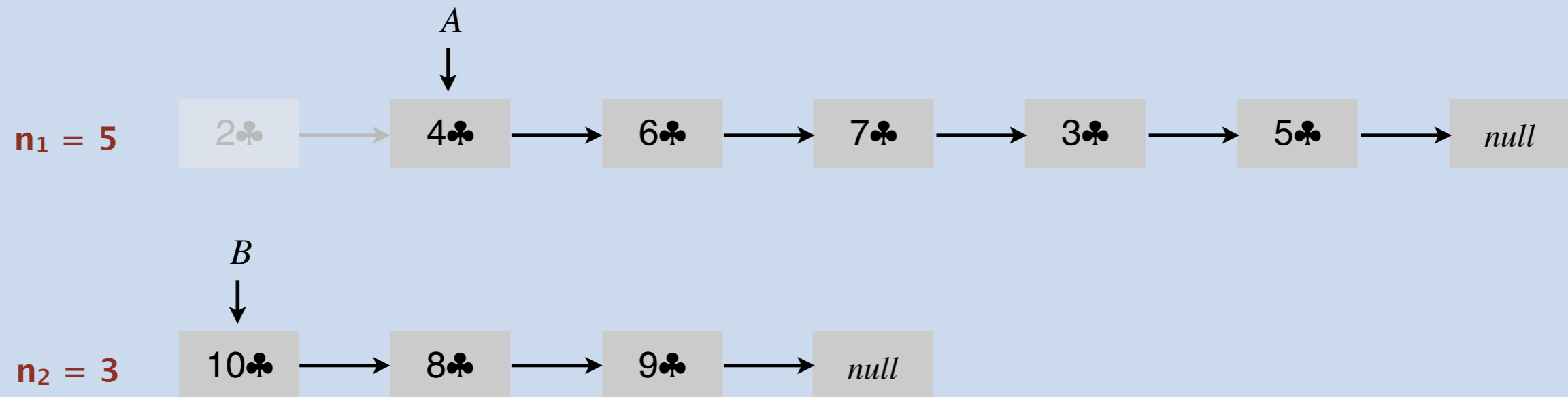


Solution. Choose first element from A with probability $n_1 / (n_1 + n_2)$;
choose first element from B with probability $n_2 / (n_1 + n_2)$; repeat.

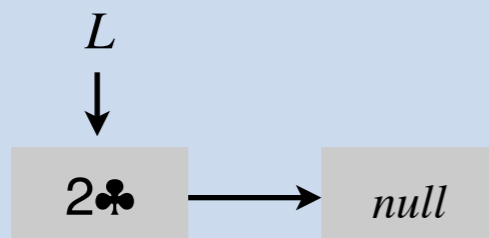


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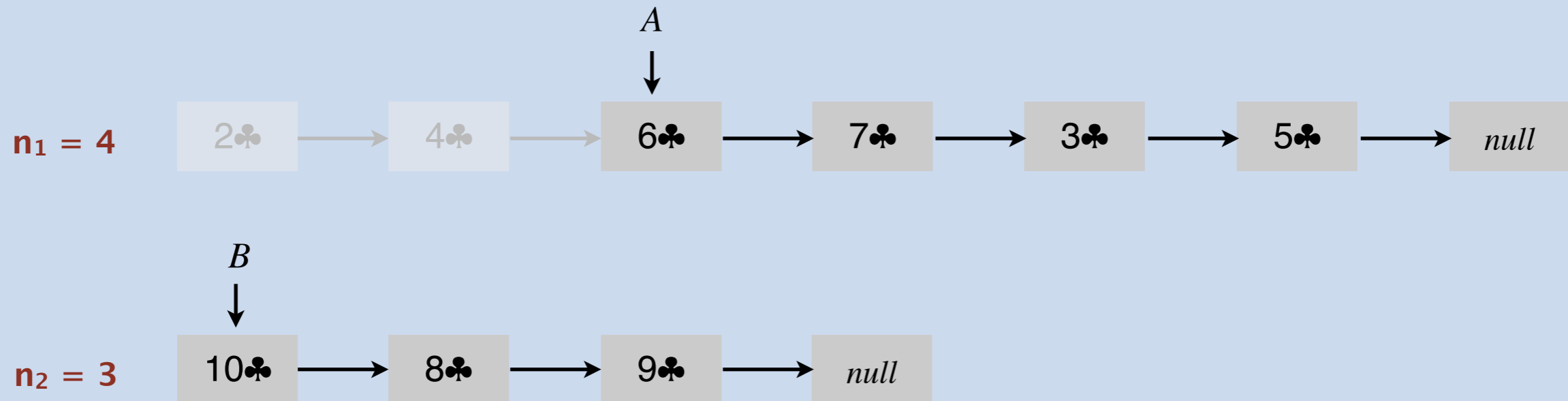


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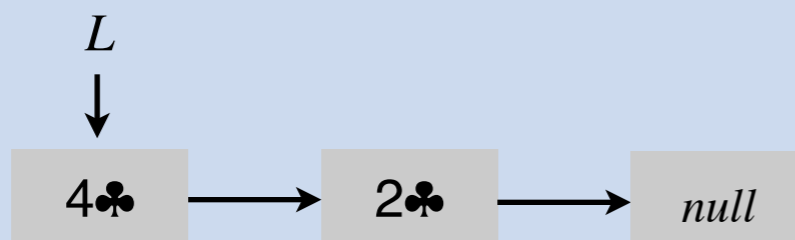
Merging two shuffled linked lists

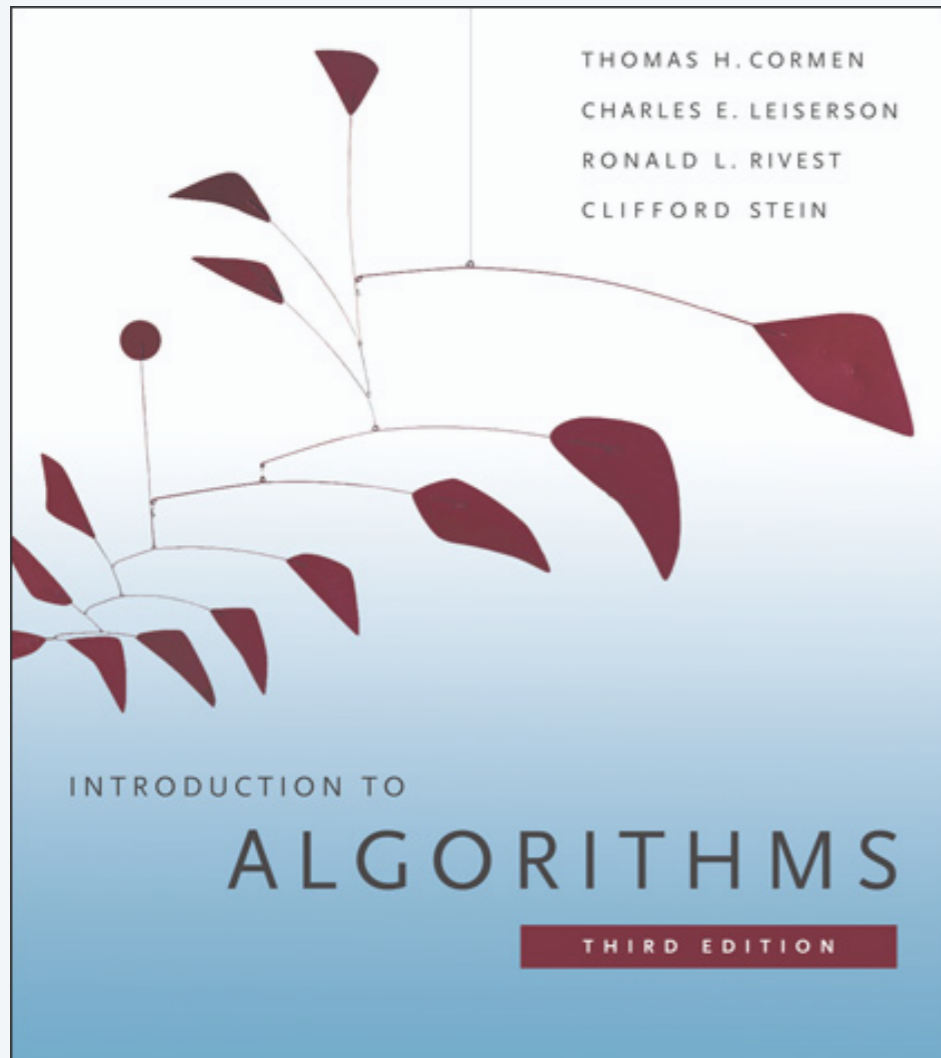
Problem. Given two shuffled linked lists, create a shuffled linked list whole.



Gilbert–Shannon–Reeds model

Solution. Choose first element from A with probability $n_1 / (n_1 + n_2)$;
choose first element from B with probability $n_2 / (n_1 + n_2)$; repeat.





SECTIONS 4.4–4.6

DIVIDE AND CONQUER II

- ▶ *mergesort*
- ▶ ***Master Theorem***
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ *counting inversions*
- ▶ *closest pair of points*
- ▶ *integer multiplication*

Divide-and-conquer recurrences

Goal. Recipe for solving common divide-and-conquer recurrences:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

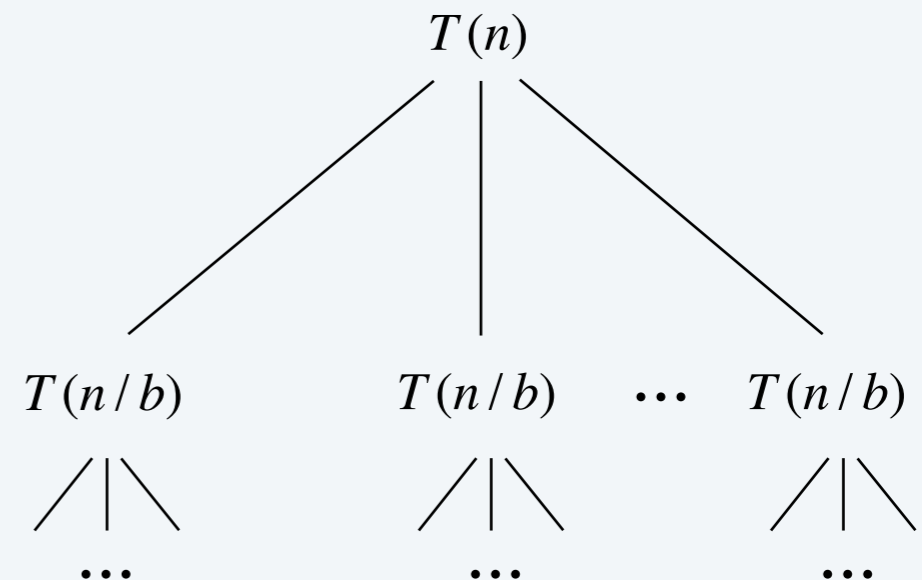
with $T(0) = 0$ and $T(1) = \Theta(1)$.

Terms.

- $a \geq 1$ is the number of subproblems.
- $b \geq 2$ is the factor by which the subproblem size decreases.
- $f(n) \geq 0$ is the work to divide and combine subproblems.

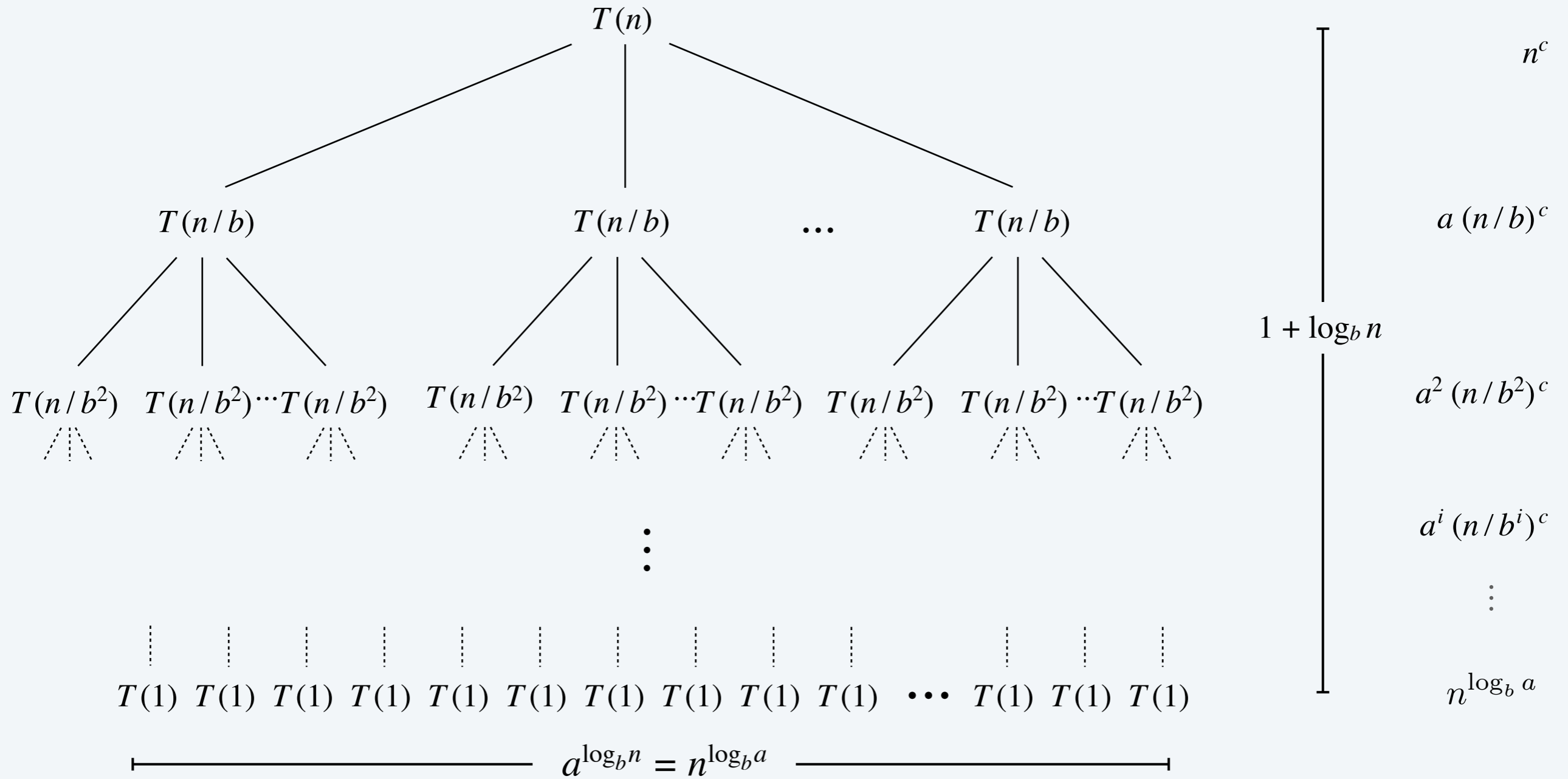
Recursion tree. [assuming n is a power of b]

- a = branching factor.
- a^i = number of subproblems at level i .
- $1 + \log_b n$ levels.
- n / b^i = size of subproblem at level i .



Divide-and-conquer recurrences: recursion tree

Suppose $T(n)$ satisfies $T(n) = a T(n/b) + n^c$ with $T(1) = 1$, for n a power of b .



$$r = a / b^c \quad T(n) = n^c \sum_{i=0}^{\log_b n} r^i$$

Divide-and-conquer recurrences: recursion tree analysis

Suppose $T(n)$ satisfies $T(n) = a T(n/b) + n^c$ with $T(1) = 1$, for n a power of b .

Let $r = a/b^c$. Note that $r < 1$ iff $c > \log_b a$.

$$T(n) = n^c \sum_{i=0}^{\log_b n} r^i = \begin{cases} \Theta(n^c) & \text{if } r < 1 & c > \log_b a & \longleftarrow & \text{cost dominated} \\ & & & & \text{by cost of root} \\ \Theta(n^c \log n) & \text{if } r = 1 & c = \log_b a & \longleftarrow & \text{cost evenly} \\ & & & & \text{distributed in tree} \\ \Theta(n^{\log_b a}) & \text{if } r > 1 & c < \log_b a & \longleftarrow & \text{cost dominated} \\ & & & & \text{by cost of leaves} \end{cases}$$

Geometric series.

- If $0 < r < 1$, then $1 + r + r^2 + r^3 + \dots + r^k \leq 1 / (1 - r)$.
- If $r = 1$, then $1 + r + r^2 + r^3 + \dots + r^k = k + 1$.
- If $r > 1$, then $1 + r + r^2 + r^3 + \dots + r^k = (r^{k+1} - 1) / (r - 1)$.

Divide-and-conquer recurrences: master theorem

Master theorem. Let $a \geq 1$, $b \geq 2$, and $c \geq 0$ and suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^c)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 3. If $c > \log_b a$, then $T(n) = \Theta(n^c)$.

Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.

Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.



Pf sketch.

- Prove when b is an integer and n is an exact power of b .
- Extend domain of recurrences to reals (or rationals).
- Deal with floors and ceilings. ← at most 2 extra levels in recursion tree

$$\begin{aligned} \lceil \lceil \lceil n/b \rceil / b \rceil / b \rceil &< n/b^3 + (1/b^2 + 1/b + 1) \\ &\leq n/b^3 + 2 \end{aligned}$$

Divide-and-conquer recurrences: master theorem

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Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.



Extensions.

- Can replace Θ with O everywhere.
- Can replace Θ with Ω everywhere.
- Can replace initial conditions with $T(n) = \Theta(1)$ for all $n \leq n_0$ and require recurrence to hold only for all $n > n_0$.

Divide-and-conquer recurrences: master theorem

Master theorem. Let $a \geq 1$, $b \geq 2$, and $c \geq 0$ and suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

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Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.

Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.



Ex. [Case 1] $T(n) = 3T(\lfloor n/2 \rfloor) + 5n$.

- $a = 3$, $b = 2$, $c = 1 < \log_b a = 1.5849\dots$
- $T(n) = \Theta(n^{\log_2 3}) = O(n^{1.58})$.

Divide-and-conquer recurrences: master theorem

Master theorem. Let $a \geq 1$, $b \geq 2$, and $c \geq 0$ and suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

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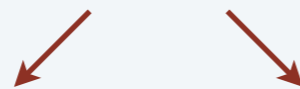
Case 3. If $c > \log_b a$, then $T(n) = \Theta(n^c)$.

Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.

Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.



ok to intermix floor and ceiling



Ex. [Case 2] $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 17n$.

- $a = 2$, $b = 2$, $c = 1 = \log_b a$.
- $T(n) = \Theta(n \log n)$.

Divide-and-conquer recurrences: master theorem

Master theorem. Let $a \geq 1$, $b \geq 2$, and $c \geq 0$ and suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^c)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 3. If $c > \log_b a$, then $T(n) = \Theta(n^c)$.

Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.

Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.



Ex. [Case 3] $T(n) = 48 T(\lfloor n/4 \rfloor) + n^3$.

- $a = 48$, $b = 4$, $c = 3 > \log_b a = 2.7924\dots$
- $T(n) = \Theta(n^3)$.

Master theorem need not apply

Gaps in master theorem.

- Number of subproblems is not a constant.

$$T(n) = nT(n/2) + n^2$$

- Number of subproblems is less than 1.

$$T(n) = \frac{1}{2}T(n/2) + n^2$$

- Work to divide and combine subproblems is not $\Theta(n^c)$.

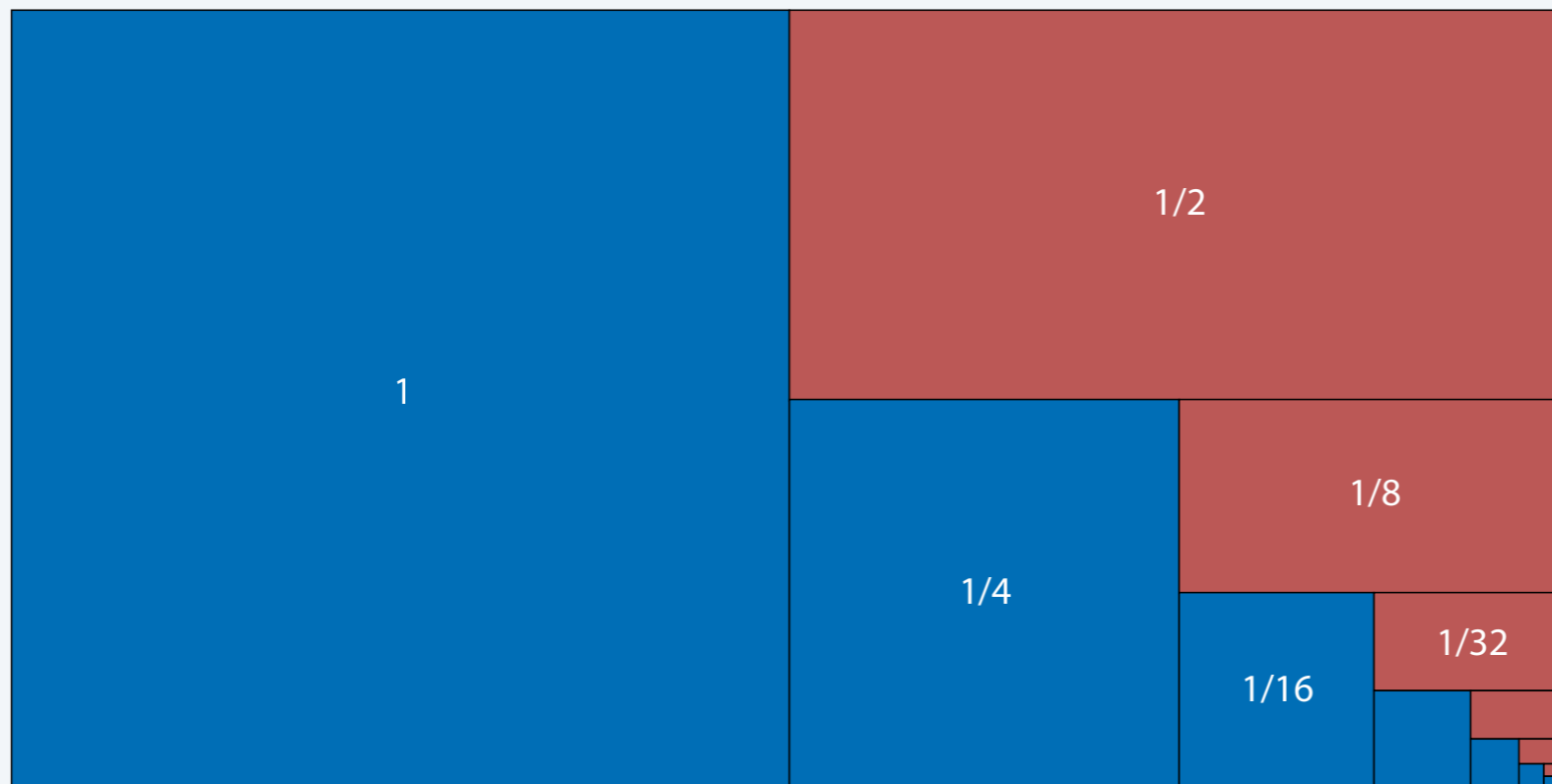
$$T(n) = 2T(n/2) + n \log n$$

Geometric series

Fact 1. If $r \neq 1$, then $1 + r + r^2 + r^3 + \dots + r^{k-1} = \frac{1 - r^k}{1 - r}$

Fact 2. If $r = 1$, then $1 + r + r^2 + r^3 + \dots + r^{k-1} = k$

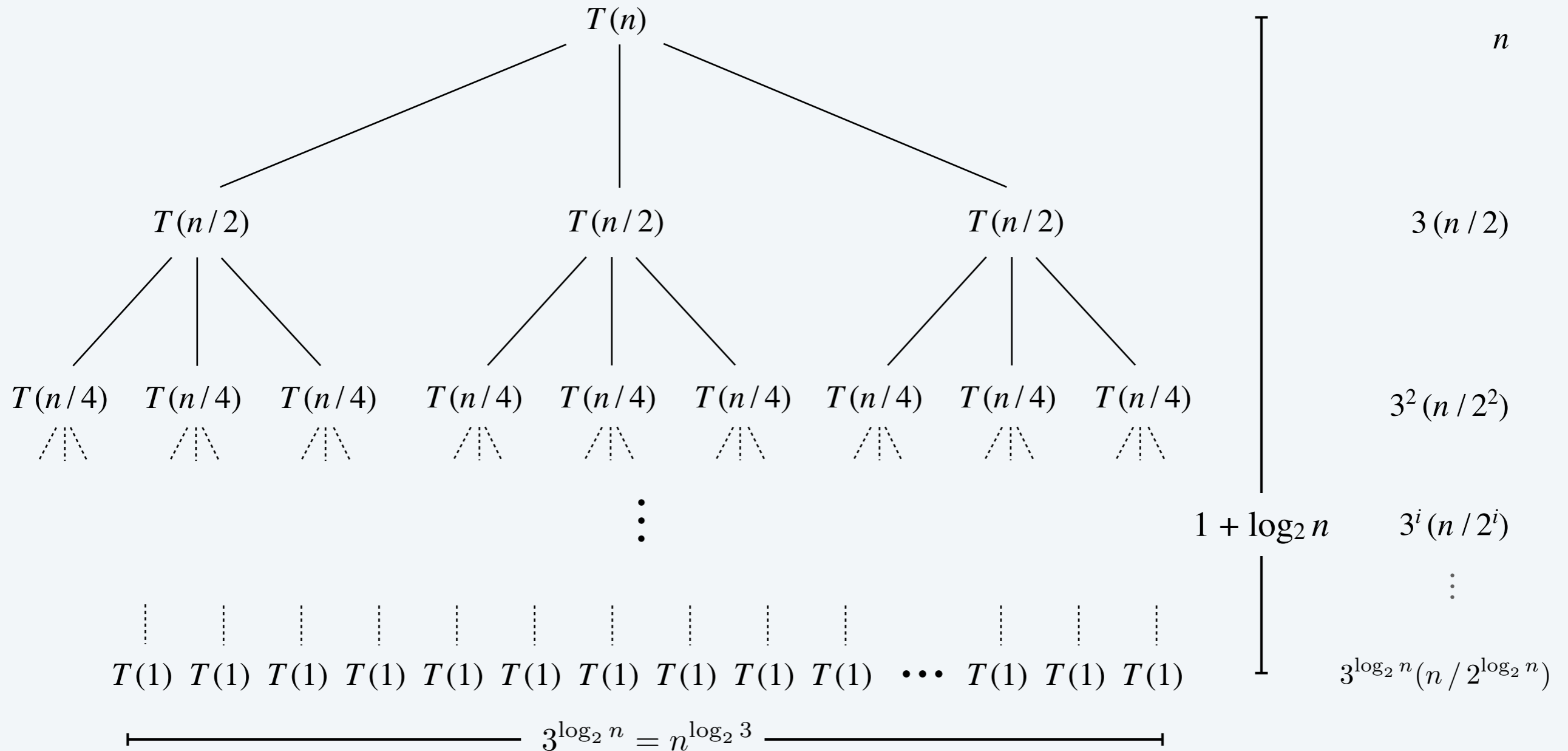
Fact 3. If $|r| < 1$, then $1 + r + r^2 + r^3 + \dots = \frac{1}{1 - r}$



$$1 + 1/2 + 1/4 + 1/8 + \dots = 2$$

Case 1: total cost dominated by cost of leaves

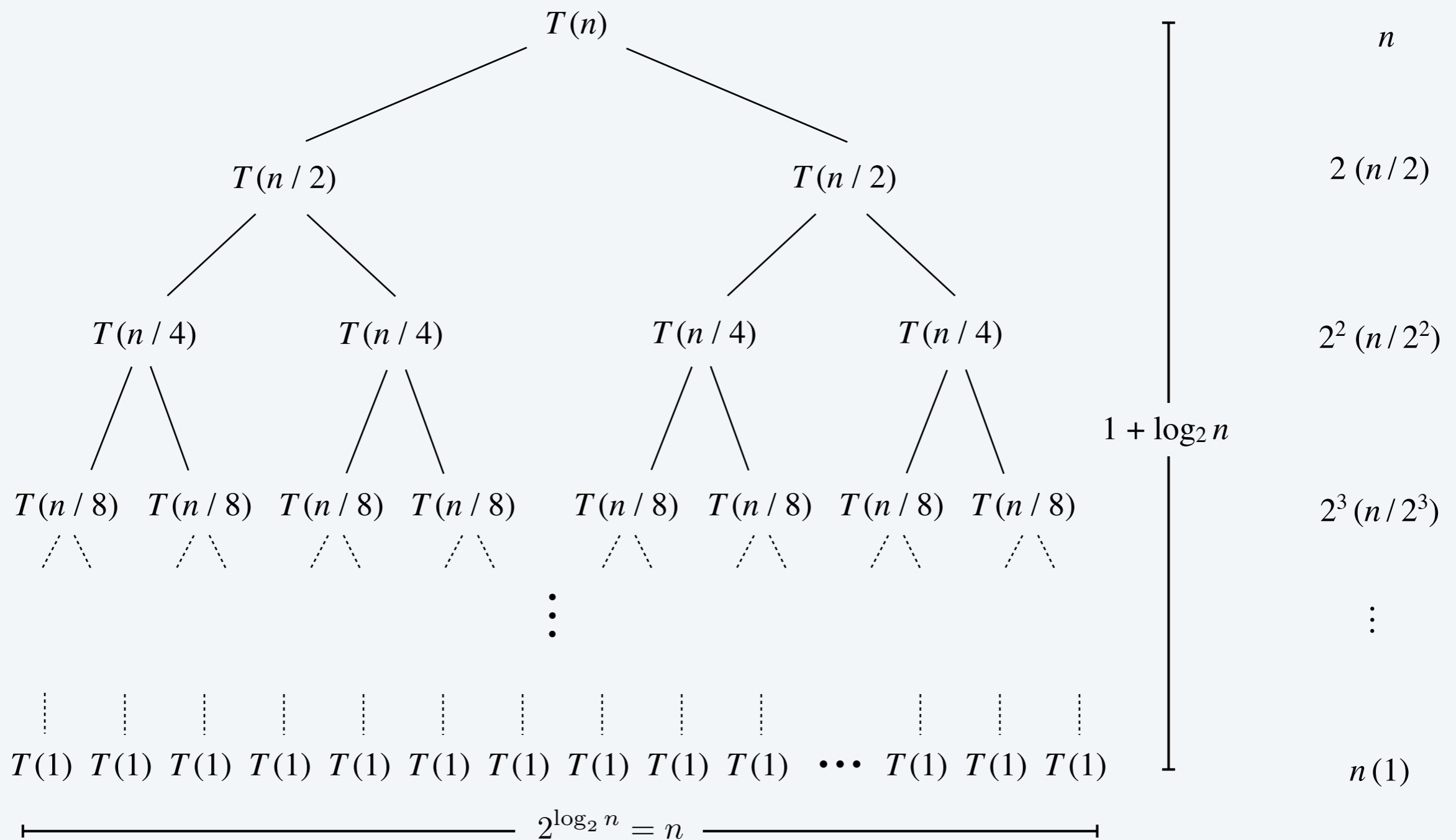
Ex 1. If $T(n)$ satisfies $T(n) = 3 T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n^{\log_2 3})$



$$r = 3/2 > 1 \quad T(n) = (1 + r + r^2 + r^3 + \dots + r^{\log_2 n}) n = \frac{r^{1+\log_2 n} - 1}{r - 1} n = 3n^{\log_2 3} - 2n$$

Case 2: total cost evenly distributed among levels

Ex 2. If $T(n)$ satisfies $T(n) = 2T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n \log n)$.

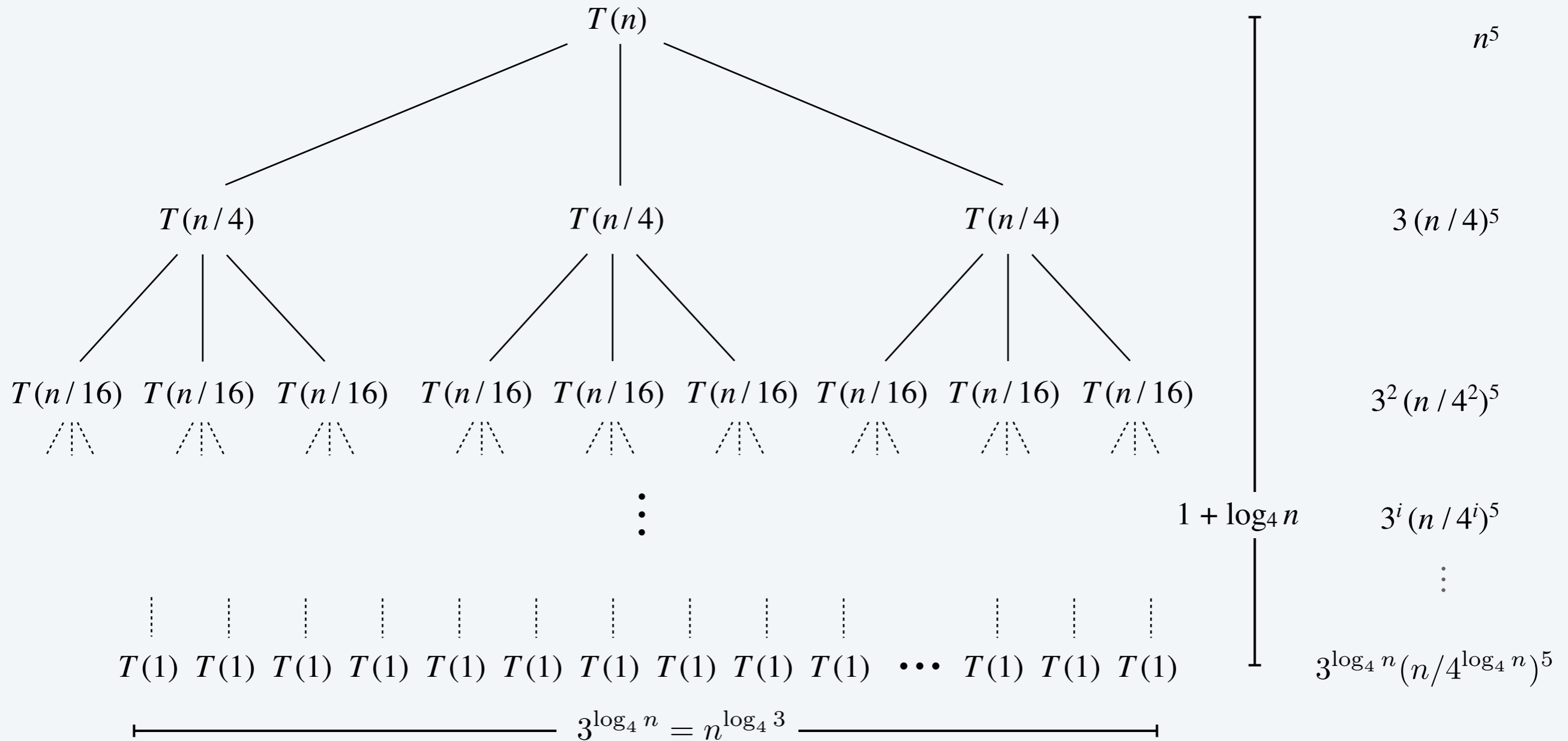


$$r = 1$$

$$T(n) = (1 + r + r^2 + r^3 + \dots + r^{\log_2 n}) n = n (\log_2 n + 1)$$

Case 3: total cost dominated by cost of root

Ex 3. If $T(n)$ satisfies $T(n) = 3 T(n / 4) + n^5$, with $T(1) = 1$, then $T(n) = \Theta(n^5)$.



$$r = 3 / 4^5 < 1 \quad n^5 \leq T(n) \leq (1 + r + r^2 + r^3 + \dots) n^5 \leq \frac{1}{1 - r} n^5$$

Master theorem (divide-and-conquer)

Master theorem. Suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

Ex. $T(n) = 3 T(n/2) + 5n$.

- $a = 3$, $b = 2$, $\log_b a = 1.58\dots$, $f(n) = 5n$, $\varepsilon = 0.58$.
- $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$.

Master theorem (divide-and-conquer)

Master theorem. Suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.

Ex. $T(n) = 2T(\lceil n/2 \rceil) + 17n$.

- $a = 2, b = 2, \log_b a = 1, f(n) = 17n$.
- $T(n) = \Theta(n \log n)$.

Master theorem (divide-and-conquer)


Master theorem. Suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$.

“regularity condition”
holds if $f(n) = \Theta(n^k)$



Ex. $T(n) = 3 T(n/4) + n^5$.

- $a = 3$, $b = 4$, $\log_b a = 0.7924\dots$, $f(n) = n^5$.
- Regularity condition: $3(n/4)^5 \leq c n^5$ for $c = 3/4^5 < 1$.
- $T(n) = \Theta(f(n)) = \Theta(n^5)$.

Master theorem (divide-and-conquer)

Master theorem. Suppose that $T(n)$ is a function on the non-negative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$


with $T(0) = 0$ and $T(1) = \Theta(1)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then,

Case 1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

Case 2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.

Case 3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$.

Pf sketch.

- Use recursion tree to sum up terms (assuming n is an exact power of b).
- Three cases for geometric series.
- Deal with floors and ceilings.  at most 2 extra levels in recursion tree



Consider the following recurrence. Which case of the master theorem?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 3T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- A.** Case 3: $T(n) = \Theta(n)$.
- B.** Case 2: $T(n) = \Theta(n \log n)$.
- C.** Case 1: $T(n) = \Theta(n^{\log_2 3}) = O(n^{1.585})$.
- D.** Master theorem not applicable.



Consider the following recurrence. Which case of the master theorem?

$$T(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n & \text{if } n > 1 \end{cases}$$

- A.** Case 1: $T(n) = \Theta(n)$.
- B.** Case 2: $T(n) = \Theta(n \log n)$.
- C.** Case 3: $T(n) = \Theta(n)$.
- D.** Master theorem not applicable.

Master theorem need not apply

Gaps in master theorem.

- Number of subproblems is not a constant.

$$T(n) = nT(n/2) + n^2$$

- Number of subproblems is less than 1.

$$T(n) = \frac{1}{2}T(n/2) + n^2$$

- No polynomial separation between $f(n)$ and $n^{\log_b a}$.

$$T(n) = 2T(n/2) + n \log n$$

- $f(n)$ is not positive.

$$T(n) = 2T(n/2) - n^2$$

- Regularity condition does not hold.

$$T(n) = T(n/2) + n(2 - \cos n)$$

Akra–Bazzi theorem

Theorem. [Akra–Bazzi 1998] Given constants $a_i > 0$ and $0 < b_i < 1$, functions $|h_i(n)| = O(n / \log^2 n)$ and $g(n) = O(n^c)$. If $T(n)$ satisfies the recurrence:

$$T(n) = \sum_{i=1}^k a_i T(b_i n + h_i(n)) + g(n)$$

a_i subproblems
of size $b_i n$

small perturbation to handle
floors and ceilings



then, $T(n) = \Theta \left(n^p \left(1 + \int_1^n \frac{g(u)}{u^{p+1}} du \right) \right)$, where p satisfies $\sum_{i=1}^k a_i b_i^p = 1$

Ex. $T(n) = T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5 n$, with $T(0) = 0$ and $T(1) = 0$.

- $a_1 = 1, b_1 = 1/5, a_2 = 1, b_2 = 7/10 \Rightarrow p = 0.83978... < 1$.
- $h_1(n) = \lfloor n/5 \rfloor - n/5, h_2(n) = 3/10 n - 3\lfloor n/10 \rfloor$.
- $g(n) = 11/5 n \Rightarrow T(n) = \Theta(n)$.



Consider the following recurrence. Which case of the master theorem?

$$T(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n & \text{if } n > 1 \end{cases}$$

- A.** Case 1: $T(n) = \Theta(n)$.
- B.** Case 2: $T(n) = \Theta(n \log n)$.
- C.** Case 3: $T(n) = \Theta(n)$.
- D.** Master theorem not applicable.

Recurrence Relation of Quickselect

What is the recurrence relation of Quickselect?

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- Master Theorem:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

Recurrence Relation of Quickselect

What is the recurrence relation of Quickselect?

- Master Theorem:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

- a is the number of subproblems
- b is the factor by which the subproblem size decreases
- $f(n)$ is the work to divide and combine subproblems
- $r = a / b^c$
- $c ? \log_b a$

Recurrence Relation of Quickselect

What is the recurrence relation of Quickselect?

- Master Theorem:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

- a is the number of subproblems = 1
- b is the factor by which the subproblem size decreases = 2
- $f(n)$ is the work to divide and combine subproblems = $\Theta(n^c)$
- $c = 1$
- $r = a / b^c = 1 / 2^1 = 1 / 2$
- $c < \log_b a$
1 < $\log_2 1$
- $c = \log_b a$
1 = 0
- $c > \log_b a$
1 > 0

Recurrence Relation of Quickselect

What is the recurrence relation of Quickselect?

- Master Theorem:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

- a is the number of subproblems = 1
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- $f(n)$ is the work to divide and combine subproblems = $\Theta(n^c)$
- $c = 1$
- $r = a / b^c = 1 / 2^1 = 1 / 2$;
- $c ? \log_b a$
 - 1 ? $\log_2 1$
 - 1 ? 0
 - 1 > 0
- Case 3. If $c > \log_b a$, then $T(n) = \Theta(n^c)$.
- Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.
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Recurrence Relation of Quickselect

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- **Case 3.** If $c > \log_b a$, then $T(n) = \Theta(n^c)$
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- **Case 1.** If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$

Recurrence Relation of Quickselect

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- $c = 1$
- $r = a / b^c = 1 / 2^1 = 1 / 2$;
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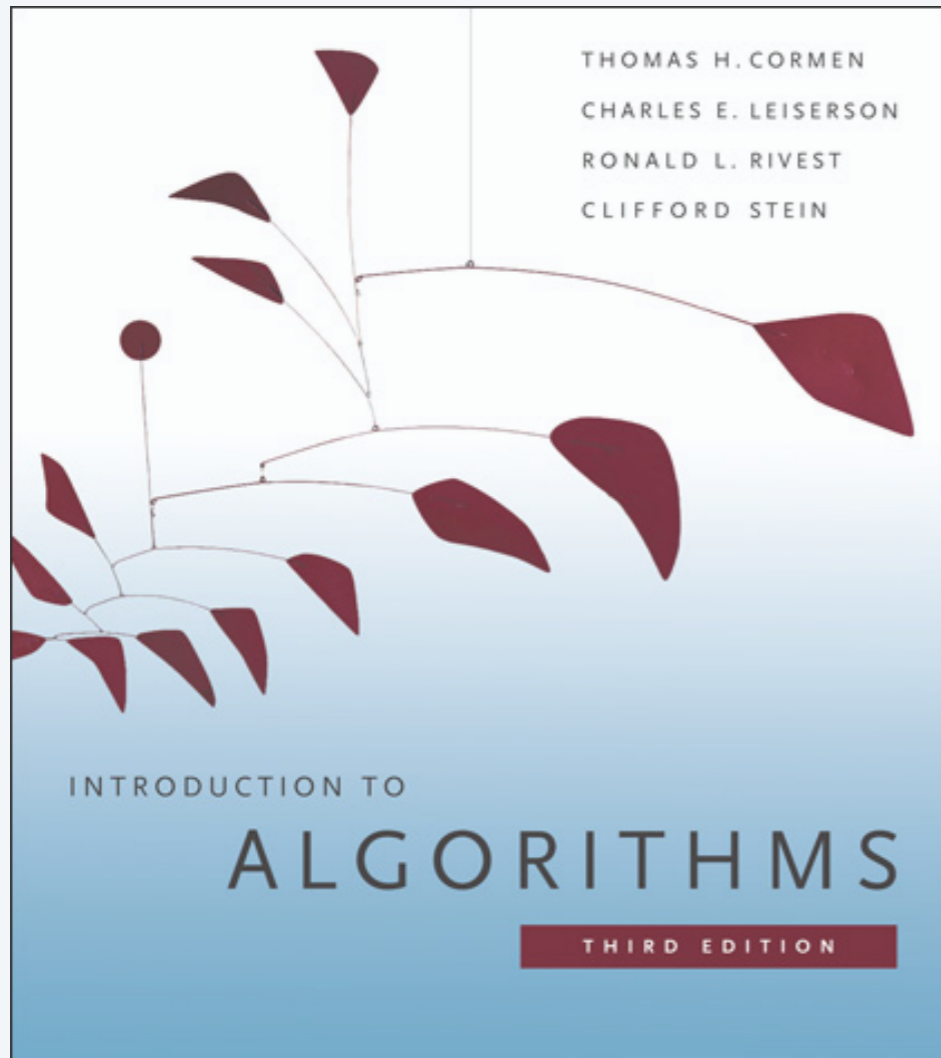
Recurrence Relation of Quickselect

What is the recurrence relation of Quickselect?

- Master Theorem:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

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- $f(n)$ is the work to divide and combine subproblems = $\Theta(n^c)$
- $c = 1$
- $r = a / b^c = 1 / 2^1 = 1 / 2$;
- $c > \log_b a$
 $1 > \log_2 1$
 $1 > 0$
 $1 > 0$
- **Case 3.** If $c > \log_b a$, then $T(n) = \Theta(n^c) = \Theta(n^1) = \Theta(n)$
- Case 2. If $c = \log_b a$, then $T(n) = \Theta(n^c \log n)$.
- Case 1. If $c < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$




SECTION 9.3

5. DIVIDE AND CONQUER

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ ***median and selection***
- ▶ *counting inversions*
- ▶ *closest pair of points*
- ▶ *integer multiplication*

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n + 1) / 2 \rfloor$.
- $O(n)$ compares for min or max.
- $O(n \log n)$ compares by sorting.
- $O(n \log k)$ compares with a binary heap.  max heap with k smallest

Applications. Order statistics; find the “top k ”; bottleneck paths, ...

Q. Can we do it with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

Randomized quickselect

- Pick a random pivot element $p \in A$.
- 3-way partition the array into L , M , and R .
- Recur in **one** subarray—the one containing the k^{th} smallest element.



QUICK-SELECT(A, k)

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY(A, p). $\longleftarrow \Theta(n)$

IF $(k \leq |L|)$ RETURN QUICK-SELECT(L, k). $\longleftarrow T(i)$

ELSE IF $(k > |L| + |M|)$ RETURN QUICK-SELECT($R, k - |L| - |M|$) $\longleftarrow T(n - i - 1)$

ELSE IF $(k = |L|)$ RETURN p .

Randomized quickselect analysis

Intuition. Split candy bar uniformly \Rightarrow expected size of larger piece is $\frac{3}{4}$.

$$T(n) \leq T(3n/4) + n \Rightarrow T(n) \leq 4n$$

not rigorous: can't assume
 $E[T(i)] \leq T(E[i])$



Def. $T(n, k)$ = expected # compares to select k^{th} smallest in array of length $\leq n$.

Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \leq 4n$.

Pf. [by strong induction on n]

- Assume true for $1, 2, \dots, n-1$.
- $T(n)$ satisfies the following recurrence:

can assume we always recur of
 larger of two subarrays since $T(n)$
 is monotone non-decreasing

$$T(n) \leq n + 1/n [2T(n/2) + \dots + 2T(n-3) + 2T(n-2) + 2T(n-1)]$$

$$\leq n + 1/n [8(n/2) + \dots + 8(n-3) + 8(n-2) + 8(n-1)]$$

$$\leq n + 1/n (3n^2)$$

$$= 4n. \quad \blacksquare$$

inductive hypothesis

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst-case linear time

Goal. Find pivot element p that divides list of n elements into two pieces so that each piece is **guaranteed** to have $\leq 7/10 n$ elements.

Q. How to find approximate median in linear time?

A. Recursively compute median of sample of $\leq 2/10 n$ elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(7/10 n) + T(2/10 n) + \Theta(n) & \text{otherwise} \end{cases}$$

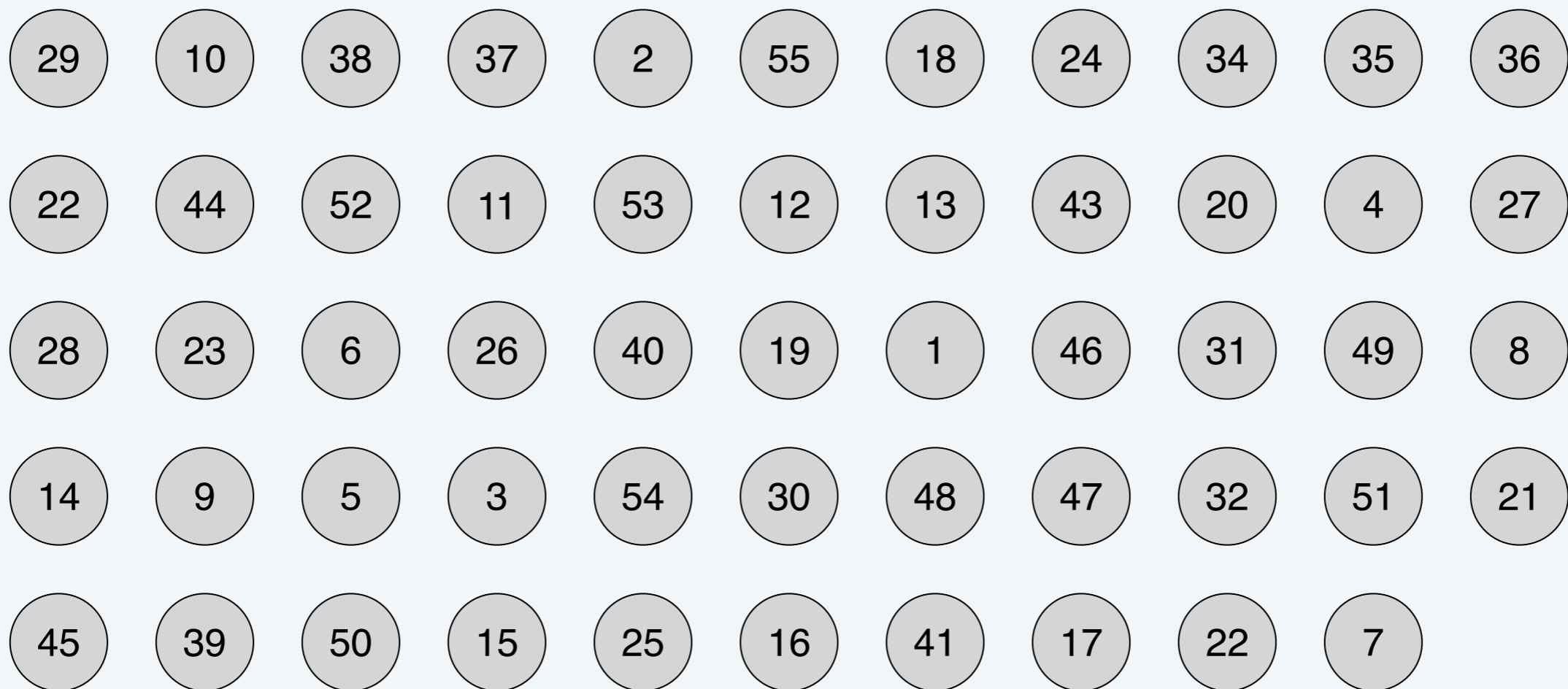
two subproblems
of different sizes!

$$\Rightarrow T(n) = \Theta(n)$$

we'll need to show this

Choosing the pivot element

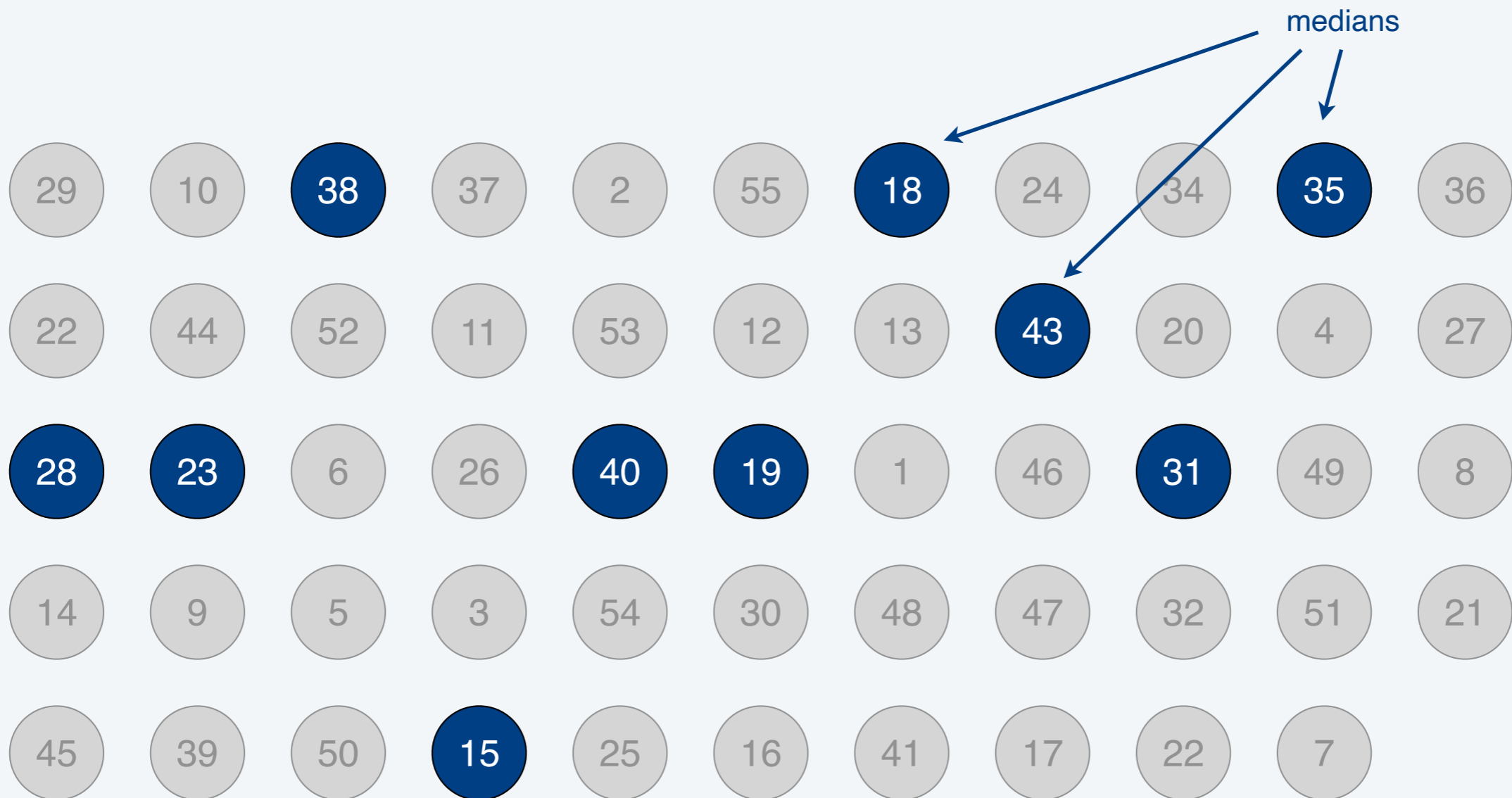
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).



$n = 54$

Choosing the pivot element

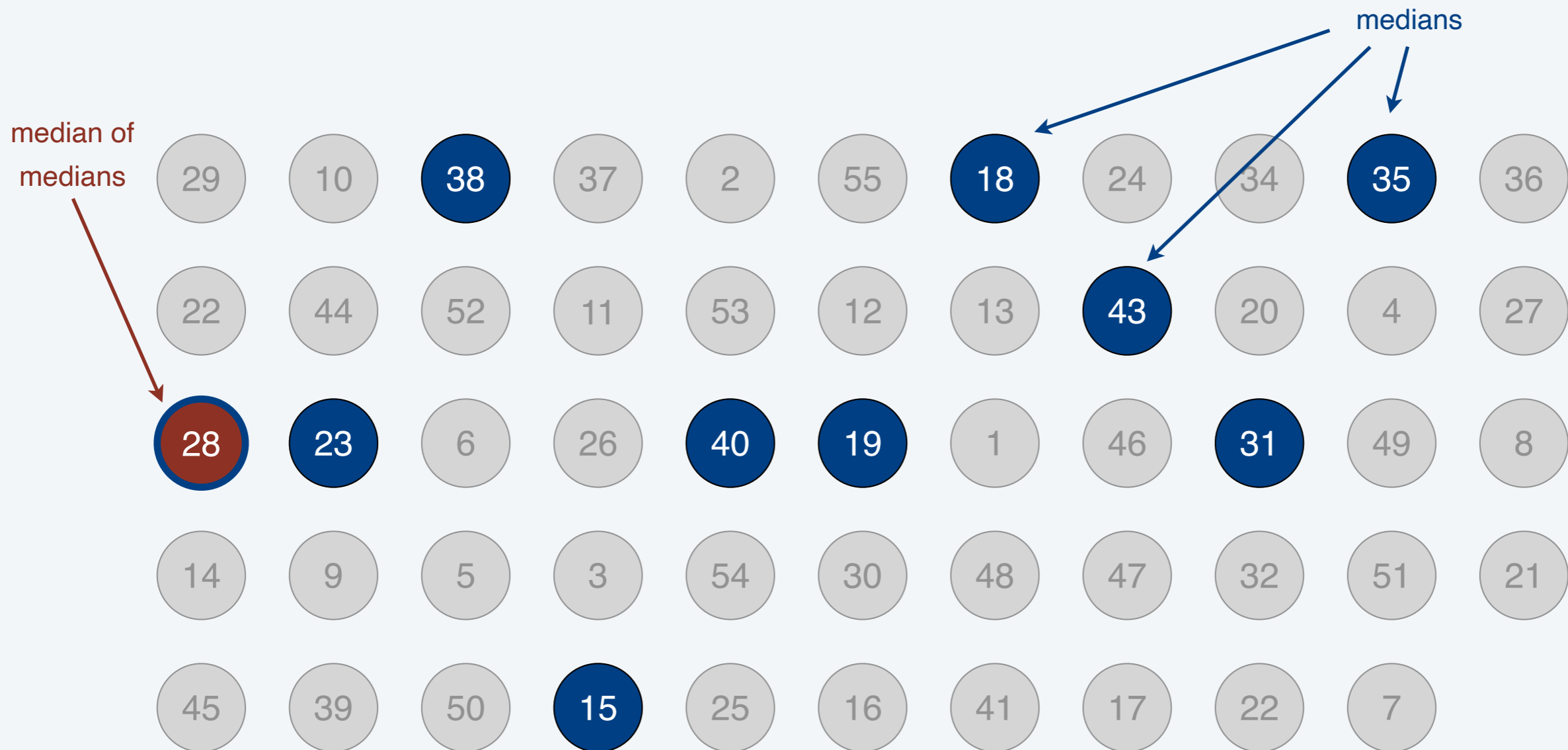
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).



$n = 54$

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n / 5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.



$n = 54$

Median-of-medians selection algorithm

MOM-SELECT(A, k)


$n \leftarrow |A|.$

IF ($n < 50$)

RETURN k^{th} smallest of element of A via mergesort.

Group A into $\lfloor n / 5 \rfloor$ groups of 5 elements each (ignore leftovers).

$B \leftarrow$ median of each group of 5.

$p \leftarrow$ **MOM-SELECT**($B, \lfloor n / 10 \rfloor$)  median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY(A, p).

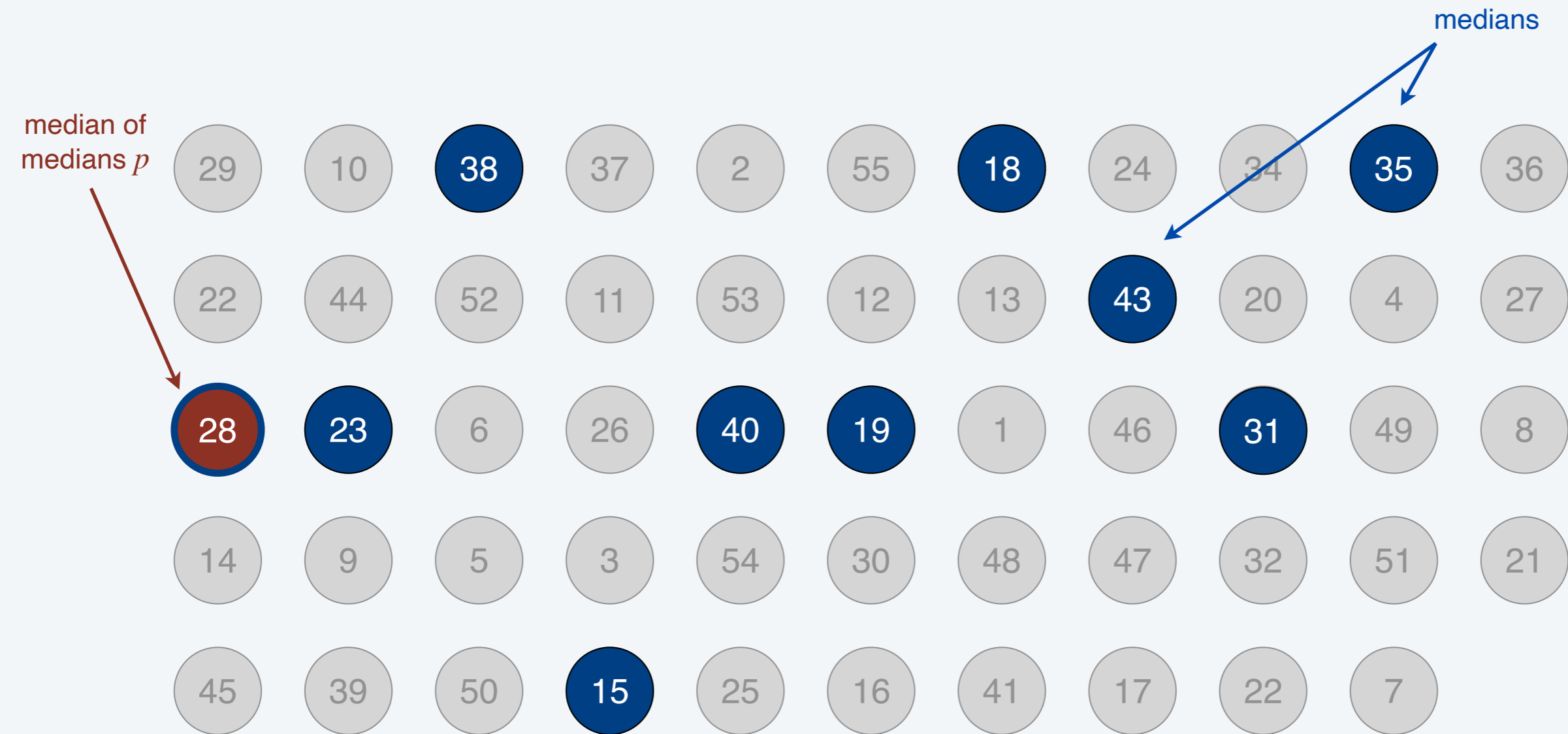
IF ($k \leq |L|$) **RETURN** **MOM-SELECT**(L, k).

ELSE IF ($k > |L| + |M|$) **RETURN** **MOM-SELECT**($R, k - |L| - |M|$)

ELSE **RETURN** p .

Analysis of median-of-medians selection algorithm

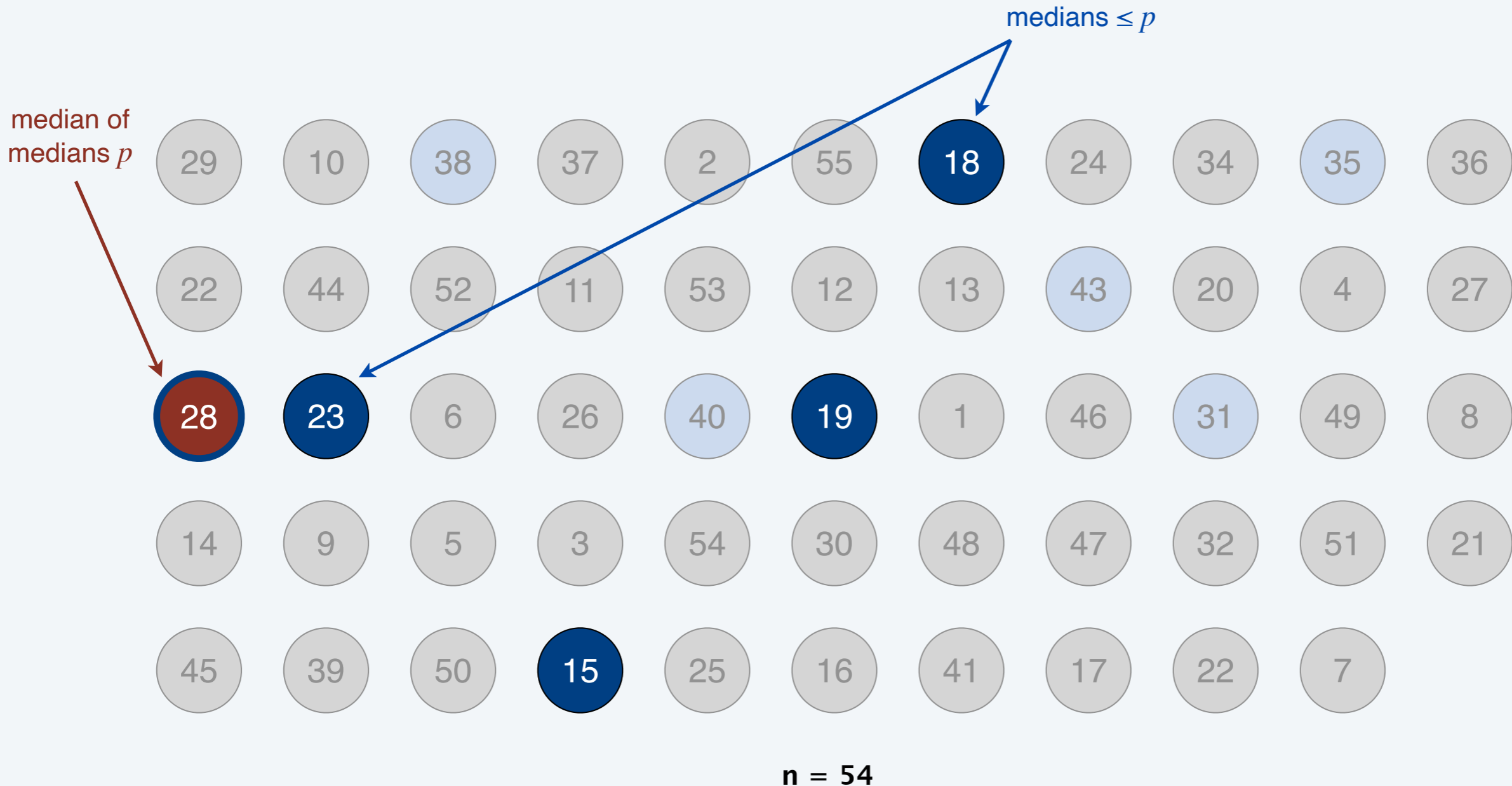
- At least half of 5-element medians $\leq p$.



$n = 54$

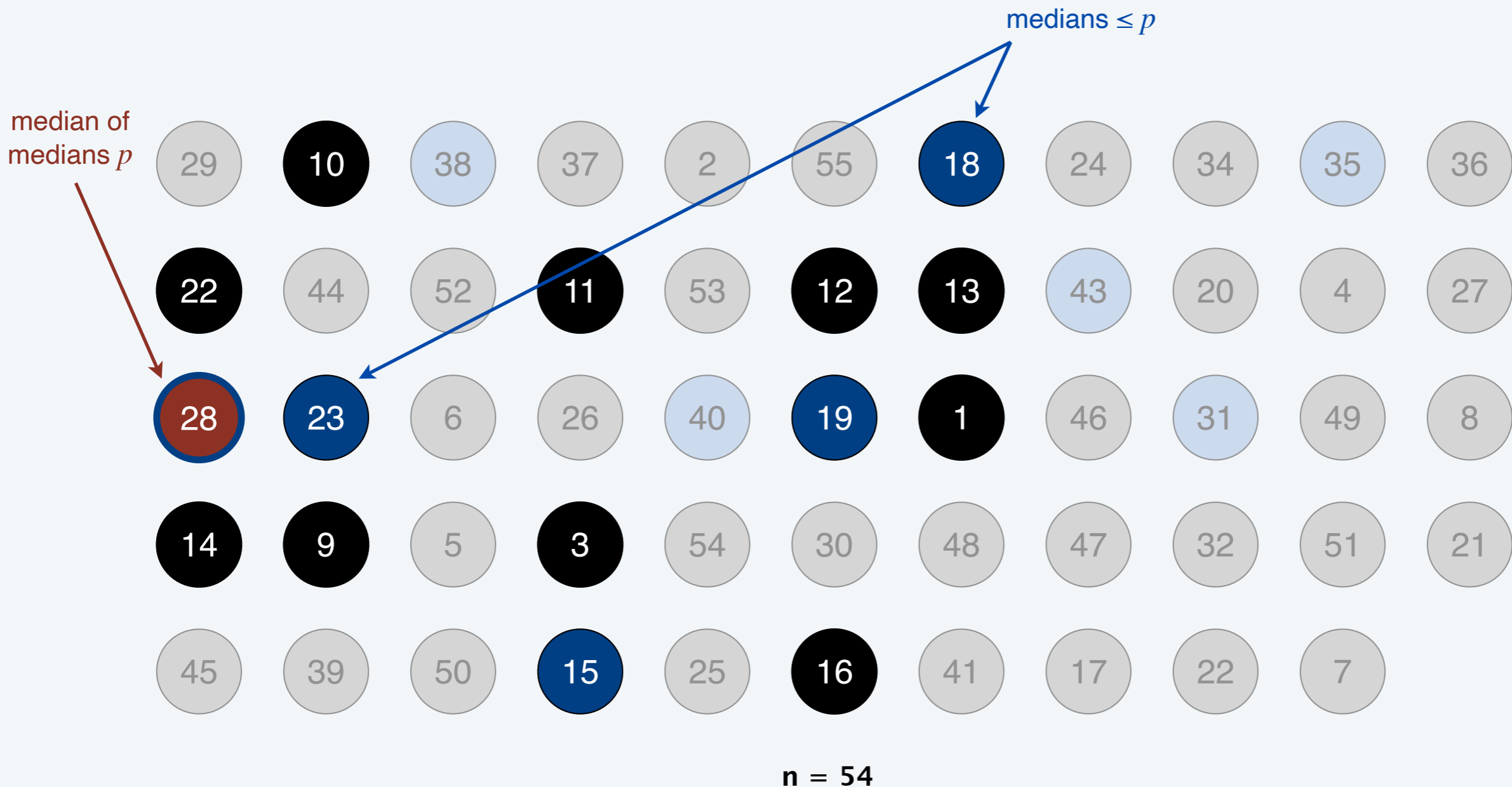
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\leq p$.



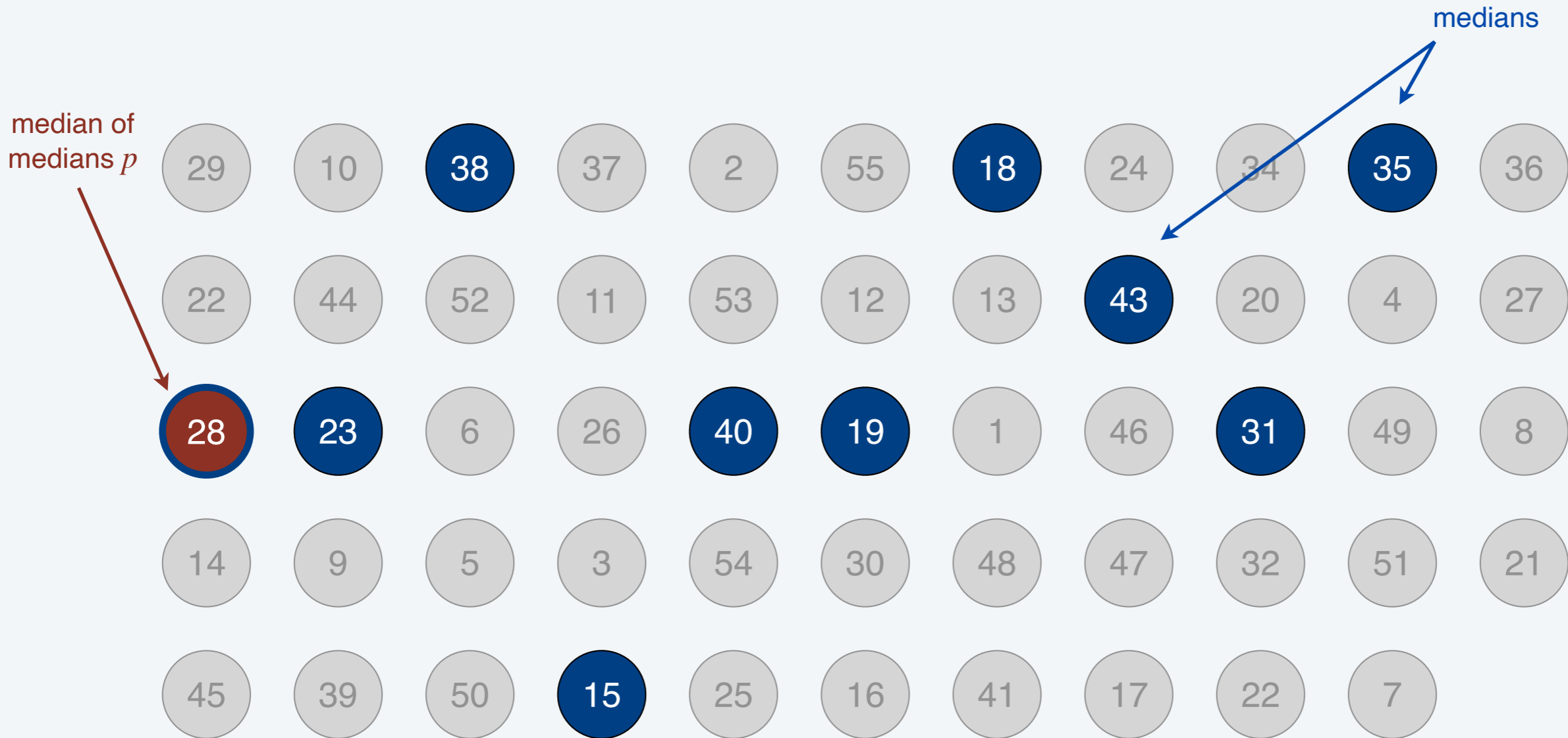
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.



Analysis of median-of-medians selection algorithm

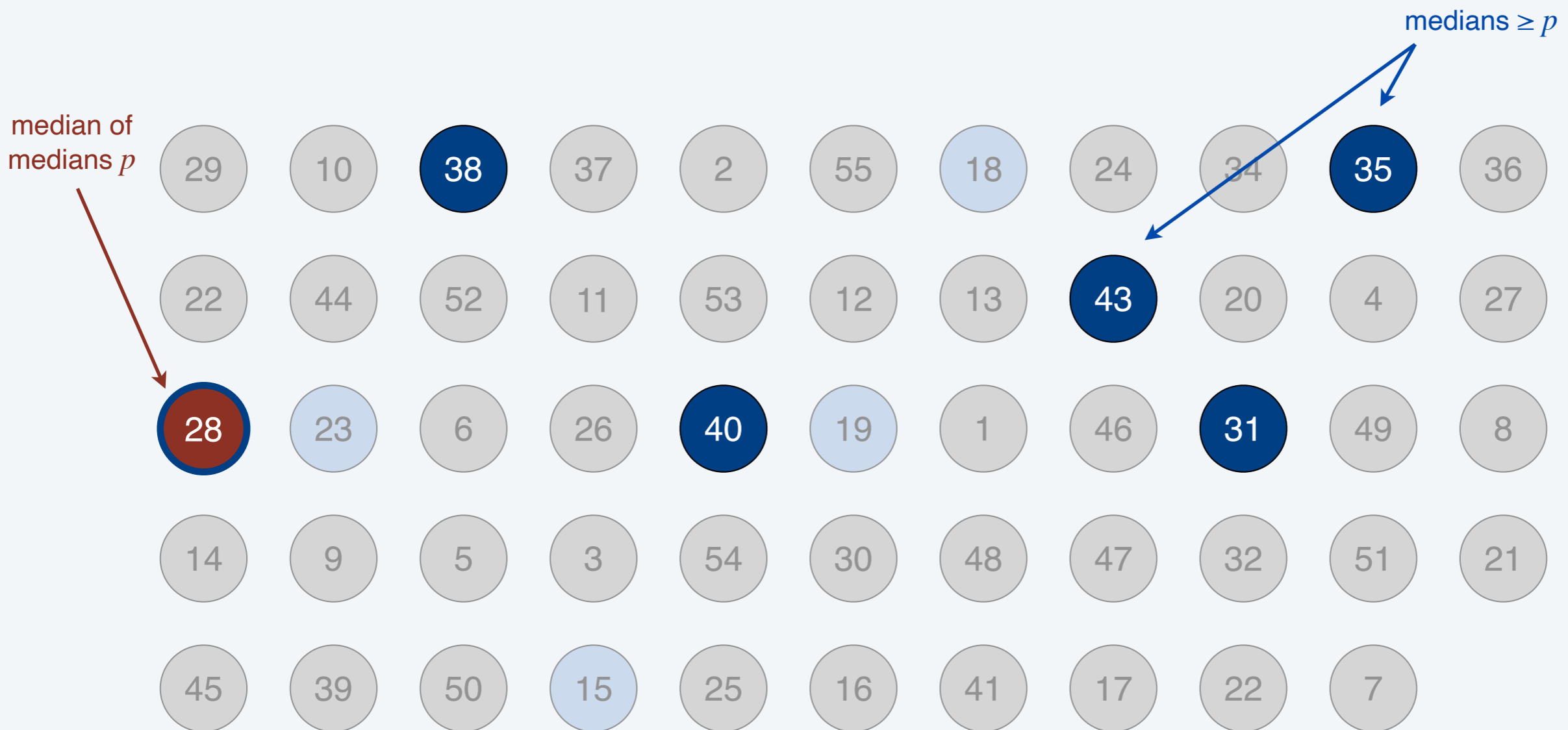
- At least half of 5-element medians $\geq p$.



n = 54

Analysis of median-of-medians selection algorithm

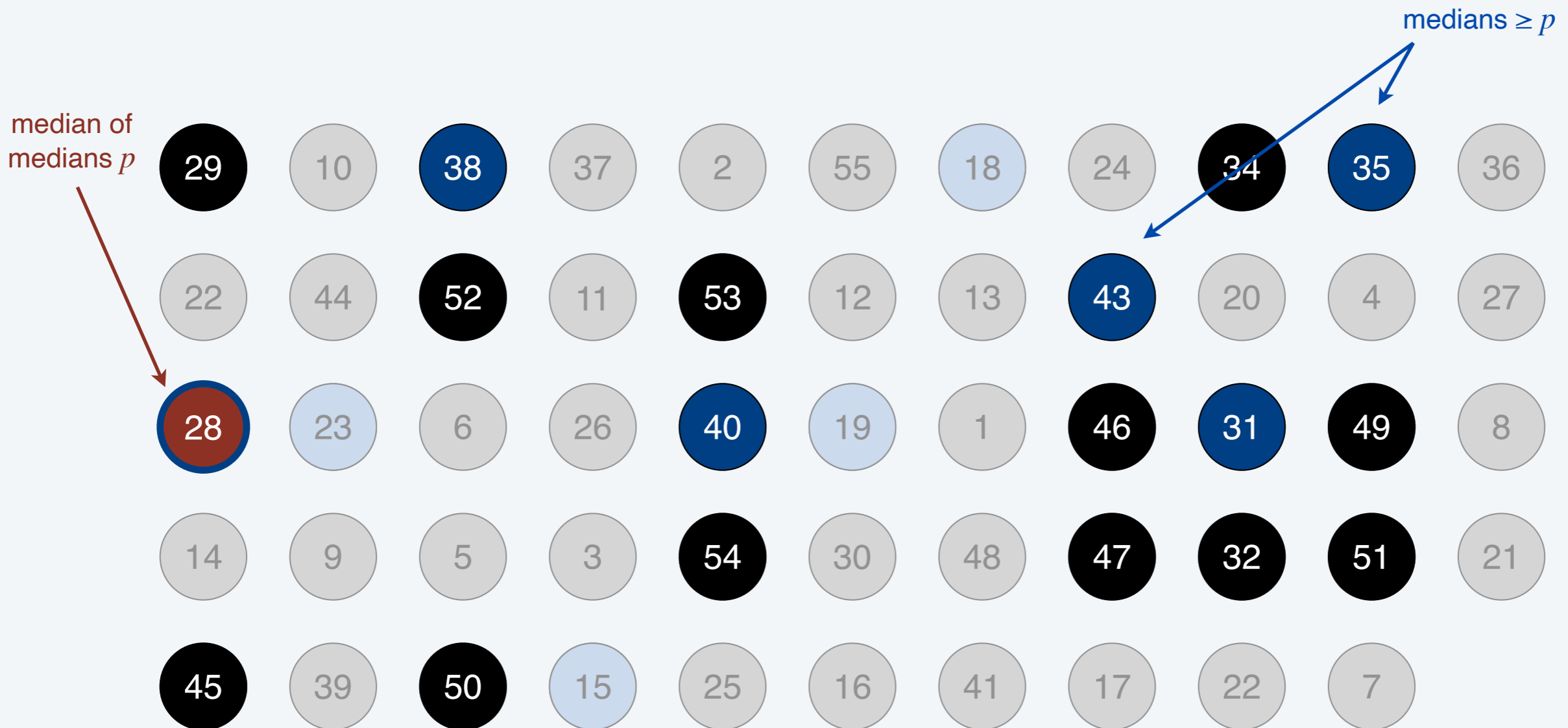
- At least half of 5-element medians $\geq p$.
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$n = 54$

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\geq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\geq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.



$n = 54$

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p .
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on any array of n elements.

$$C(n) \leq C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n$$

median of
mediansrecursive
selectcomputing median of 5
(≤ 6 compares per group)
partitioning
($\leq n$ compares)

Intuition.

- $C(n)$ is going to be at least linear in $n \Rightarrow C(n)$ is super-additive.
- Ignoring floors, this implies that $C(n) \leq C(n/5 + n - 3n/10) + 11/5 n$
 $= C(9n/10) + 11/5 n$
 $\Rightarrow C(n) \leq 22n.$

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p .
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on any array of n elements.

$$C(n) \leq C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n$$

median of
mediansrecursive
selectcomputing median of 5
(≤ 6 compares per group)
partitioning
($\leq n$ compares)

Now, let's solve given recurrence.

- Assume n is both a power of 5 and a power of 10 ?
- Prove that $C(n)$ is monotone non-decreasing.



Consider the following recurrence

$$C(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n & \text{if } n > 1 \end{cases}$$

Is $C(n)$ monotone non-decreasing?

- A. Yes, obviously.
- B. Yes, but proof is tedious.
- C. Yes, but proof is hard.
- D. No.

Median-of-medians selection algorithm recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # compares on any array of $\leq n$ elements.
- $T(n)$ is monotone non-decreasing, but $C(n)$ is not!

$$T(n) \leq \begin{cases} 6n & \text{if } n < 50 \\ \max\{ T(n-1), T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n \} & \text{if } n \geq 50 \end{cases}$$

Claim. $T(n) \leq 44n$.

Pf. [by strong induction]

- Base case: $T(n) \leq 6n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \dots, n-1$.
- Induction step: for $n \geq 50$, we have either $T(n) \leq T(n-1) \leq 44n$ or

$$\begin{aligned} T(n) &\leq T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5 n \\ \text{inductive hypothesis} \longrightarrow &\leq 44(\lfloor n/5 \rfloor) + 44(n - 3\lfloor n/10 \rfloor) + 11/5 n \\ &\leq 44(n/5) + 44n - 44(n/4) + 11/5 n \longleftarrow \text{for } n \geq 50, 3\lfloor n/10 \rfloor \geq n/4 \\ &= 44n. \quad \blacksquare \end{aligned}$$



Suppose that we divide n elements into $\lfloor n/r \rfloor$ groups of r elements each, and use the median-of-medians of these $\lfloor n/r \rfloor$ groups as the pivot. For which r is the worst-case running time of select $O(n)$?

- A. $r = 3$
- B. $r = 7$
- C. Both A and B.
- D. Neither A nor B.

Linear-time selection retrospective

Proposition. [Blum–Floyd–Pratt–Rivest–Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.

Time Bounds for Selection*

MANUEL BLUM, ROBERT W. FLOYD, VAUGHAN PRATT,
RONALD L. RIVEST, AND ROBERT E. TARJAN

Department of Computer Science, Stanford University, Stanford, California 94305

Received November 14, 1972

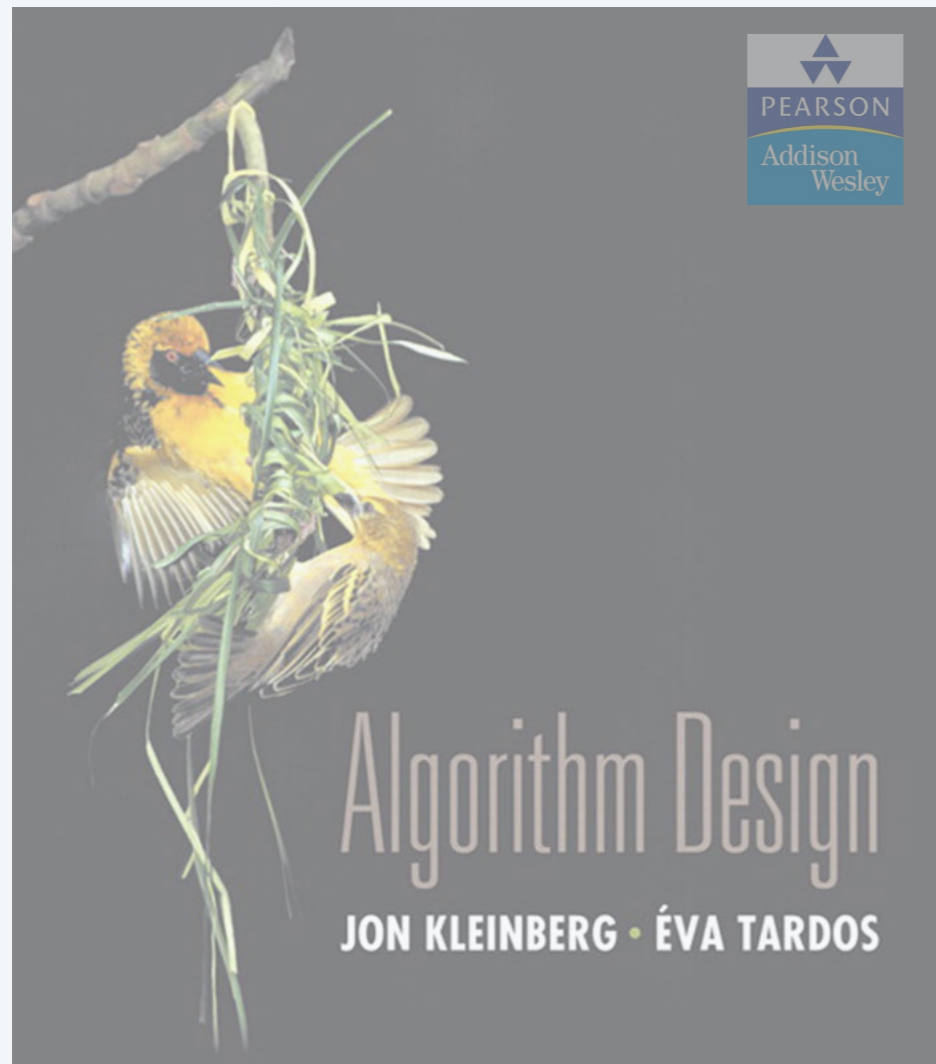
The number of comparisons required to select the i -th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm—PICK. Specifically, no more than $5.4305 n$ comparisons are ever required. This bound is improved for extreme values of i , and a new lower bound on the requisite number of comparisons is also proved.

Theory.

- Optimized version of BFPRT: $\leq 5.4305 n$ compares.
- Upper bound: [Dor–Zwick 1995] $\leq 2.95 n$ compares.
- Lower bound: [Dor–Zwick 1999] $\geq (2 + 2^{-80}) n$ compares.

Practice. BFPRT constants too large to be useful.





SECTION 5.3

5. DIVIDE AND CONQUER

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ ***counting inversions***
- ▶ *closest pair of points*
- ▶ *integer multiplication*

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of **inversions** between two rankings.

- My rank: $1, 2, \dots, n$.
- Your rank: a_1, a_2, \dots, a_n .
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.
- Collaborative filtering.
- Measuring the “sortedness” of an array.
- Sensitivity analysis of Google’s ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall’s tau distance).

Rank Aggregation Methods for the Web

Cynthia Dwork* Ravi Kumar† Moni Naor‡ D. Sivakumar§

ABSTRACT

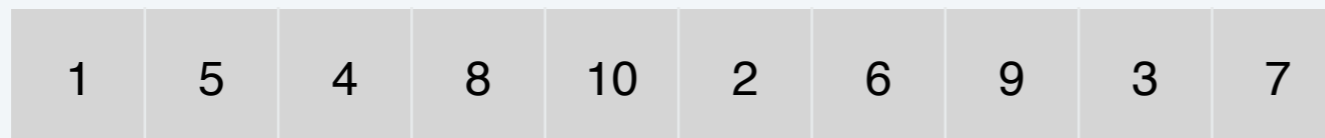
We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat “spam,” a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

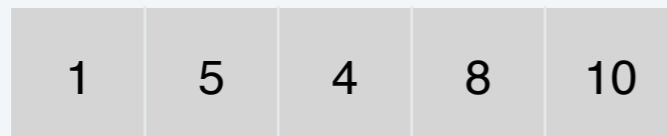
Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B .
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input



count inversions in left half A



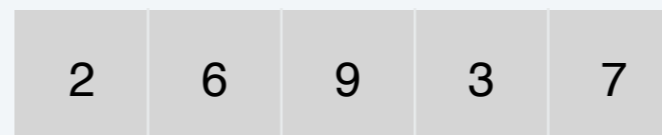
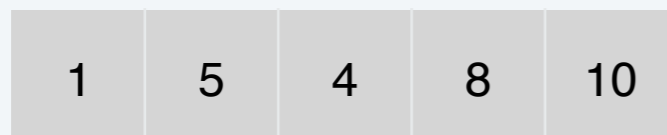
5-4

count inversions in right half B



6-3 9-3 9-7

count inversions (a, b) with $a \in A$ and $b \in B$



4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output $1 + 3 + 13 = 17$

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B .
- For each element $b \in B$,
 - binary search in A to find how elements in A are greater than b .

list A

7	10	18	3	14
---	----	----	---	----

list B

20	23	2	11	16
----	----	---	----	----

sort A

3	7	10	14	18
---	---	----	----	----

sort B

2	11	16	20	23
---	----	----	----	----

binary search to count inversions (a, b) with $a \in A$ and $b \in B$

3	7	10	14	18
---	---	----	----	----

2	11	16	20	23
---	----	----	----	----

5 2 1 0 0

Counting inversions: how to combine two subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j .
- If $a_i < b_j$, then a_i is not inverted with any element left in B .
- If $a_i > b_j$, then b_j is inverted with every element left in A .
- Append smaller element to sorted list C .



count inversions (a, b) with $a \in A$ and $b \in B$



merge to form sorted list C



Counting inversions: divide-and-conquer algorithm implementation

Input. List L .

Output. Number of inversions in L and L in sorted order.

SORT-AND-COUNT(L)

IF (list L has one element)

RETURN (0, L).

Divide the list into two halves A and B .

(r_A , A) \leftarrow **SORT-AND-COUNT**(A). $\longleftarrow T(n/2)$

(r_B , B) \leftarrow **SORT-AND-COUNT**(B). $\longleftarrow T(n/2)$

(r_{AB} , L) \leftarrow **MERGE-AND-COUNT**(A , B). $\longleftarrow \Theta(n)$

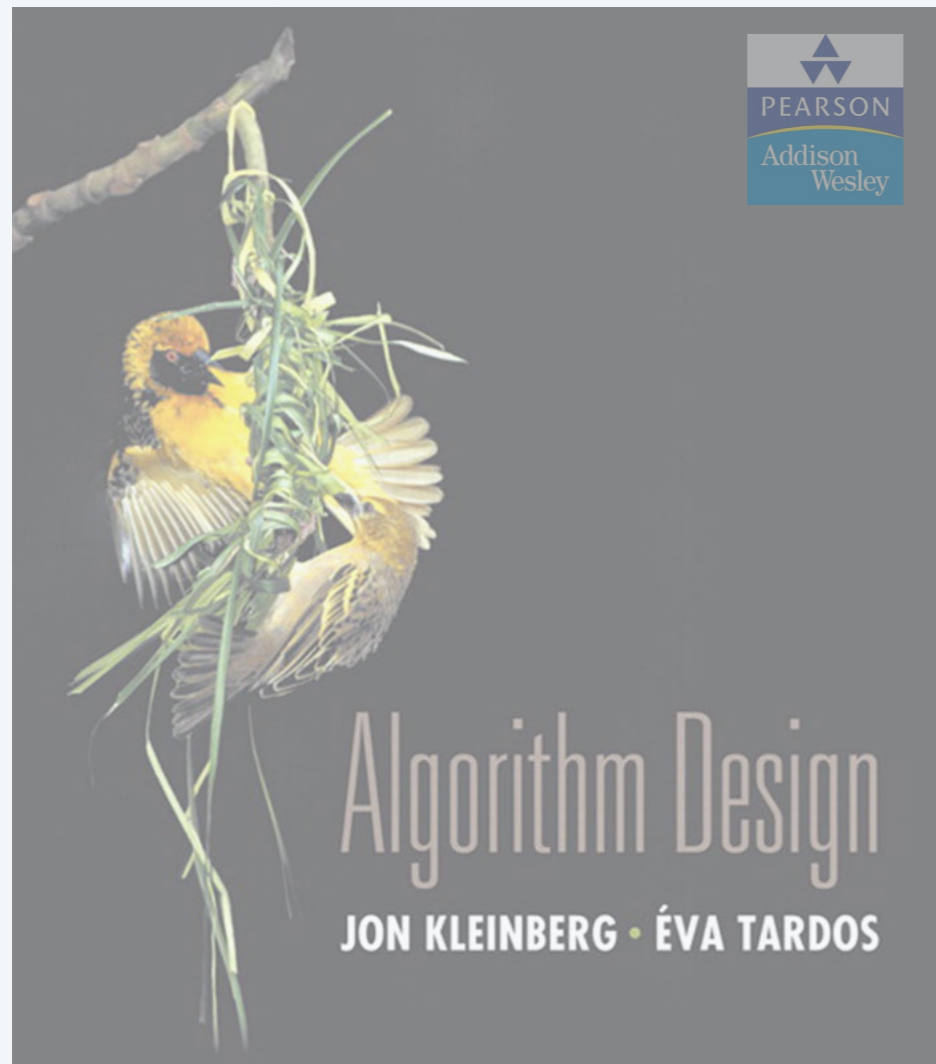
RETURN ($r_A + r_B + r_{AB}$, L).

Counting inversions: divide-and-conquer algorithm analysis

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$



SECTION 5.4

5. DIVIDE AND CONQUER

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ *counting inversions*
- ▶ ***closest pair of points***
- ▶ *integer multiplication*

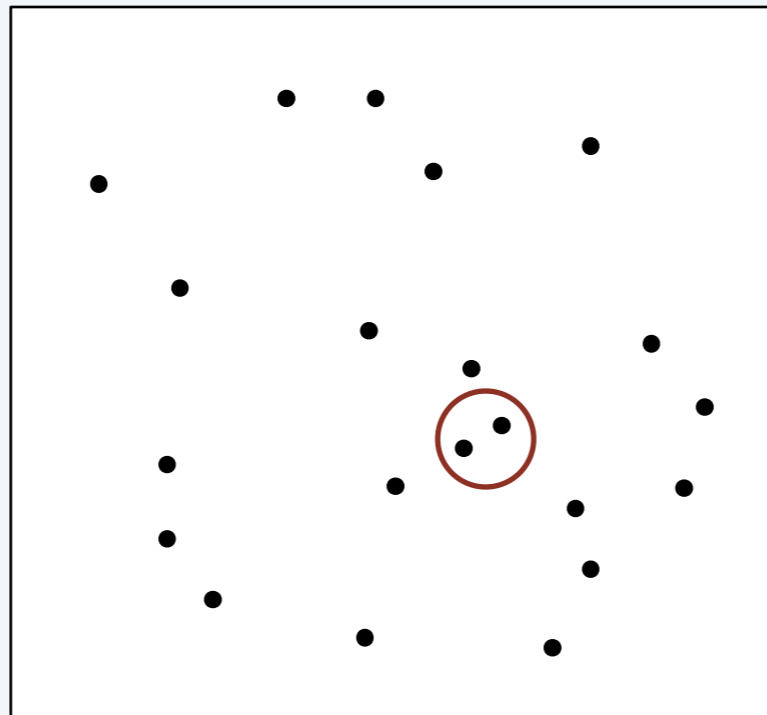
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems



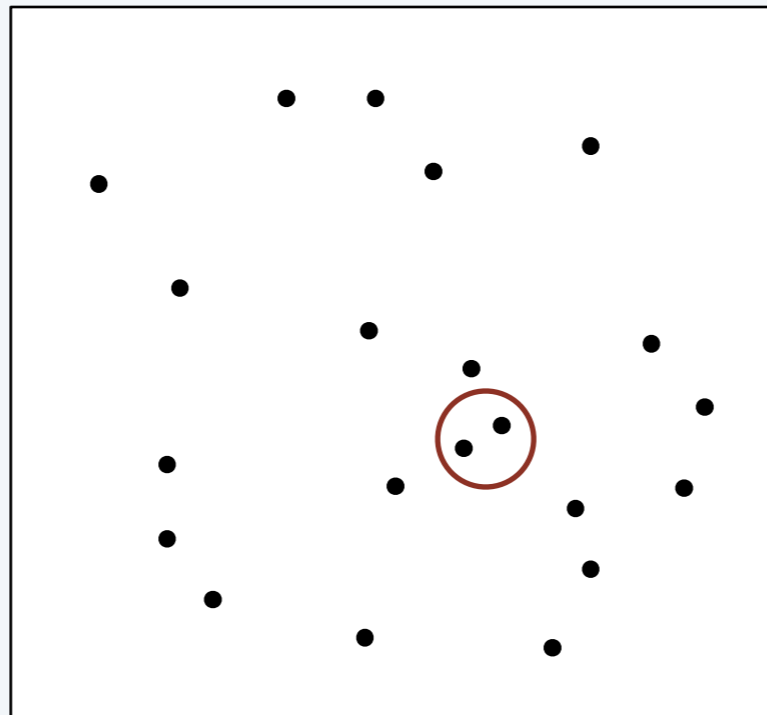
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1D version. Easy $O(n \log n)$ algorithm if points are on a line.

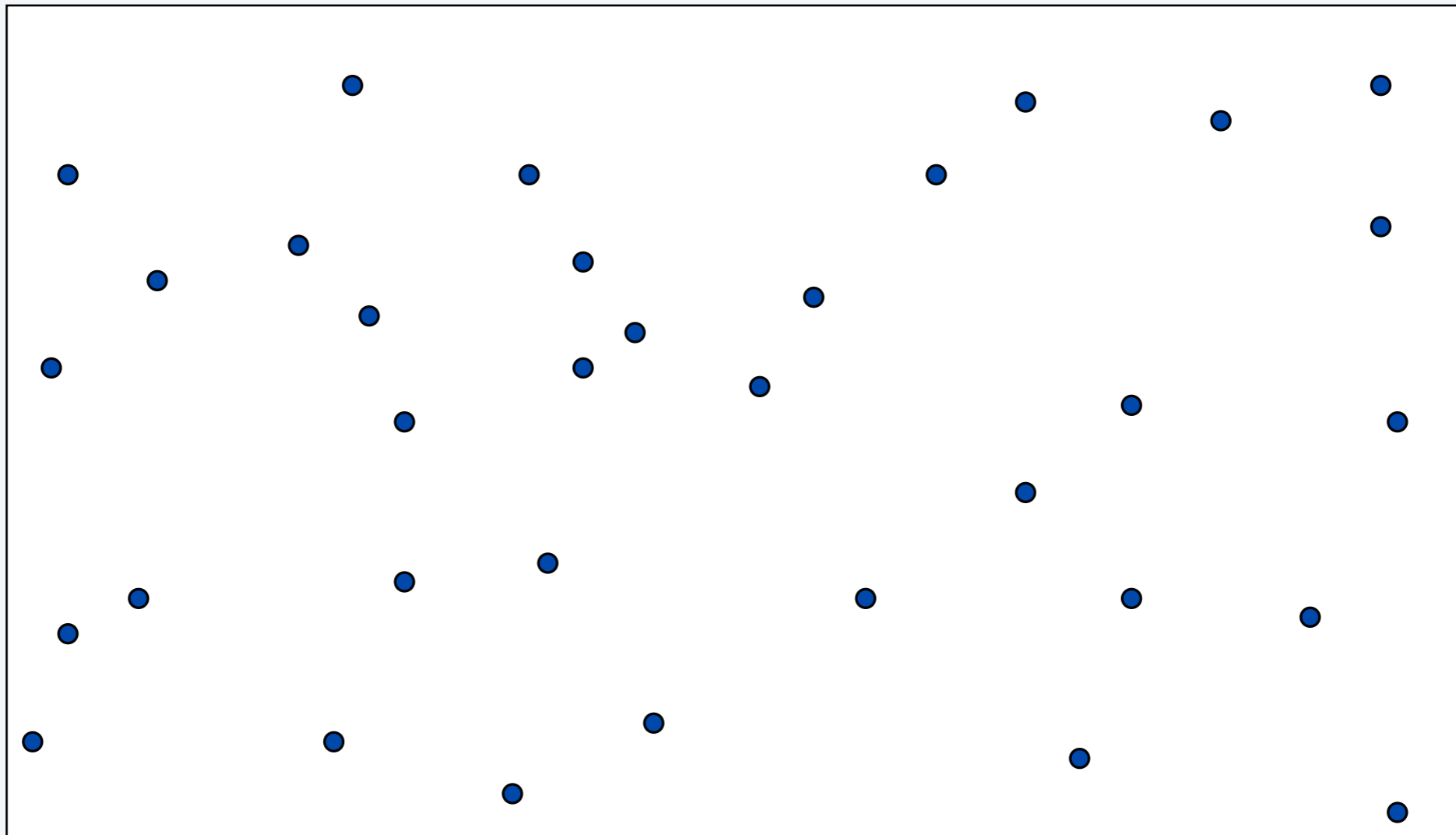
Non-degeneracy assumption. No two points have the same x -coordinate.



Closest pair of points: first attempt

Sorting solution.

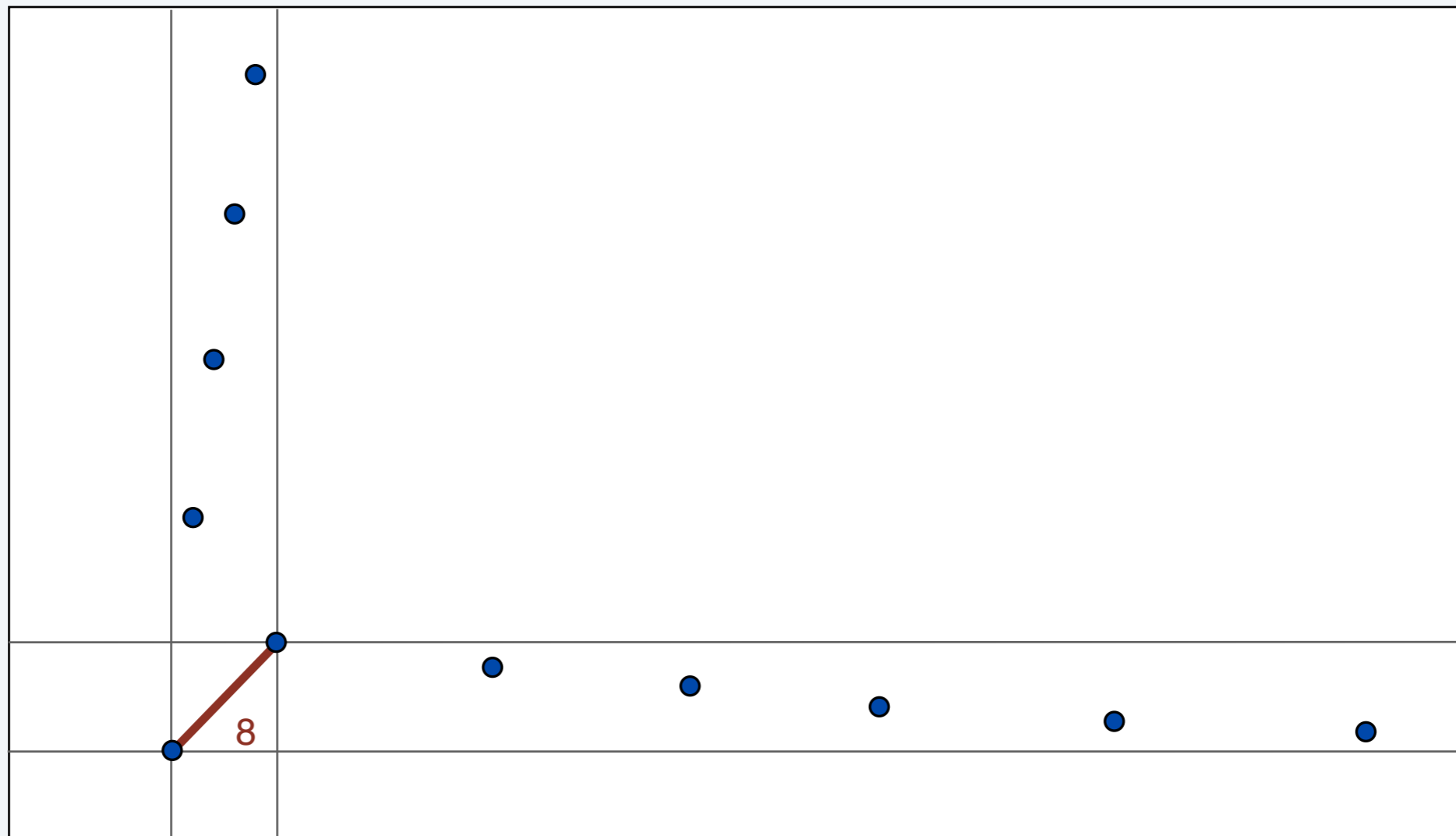
- Sort by x -coordinate and consider nearby points.
- Sort by y -coordinate and consider nearby points.



Closest pair of points: first attempt

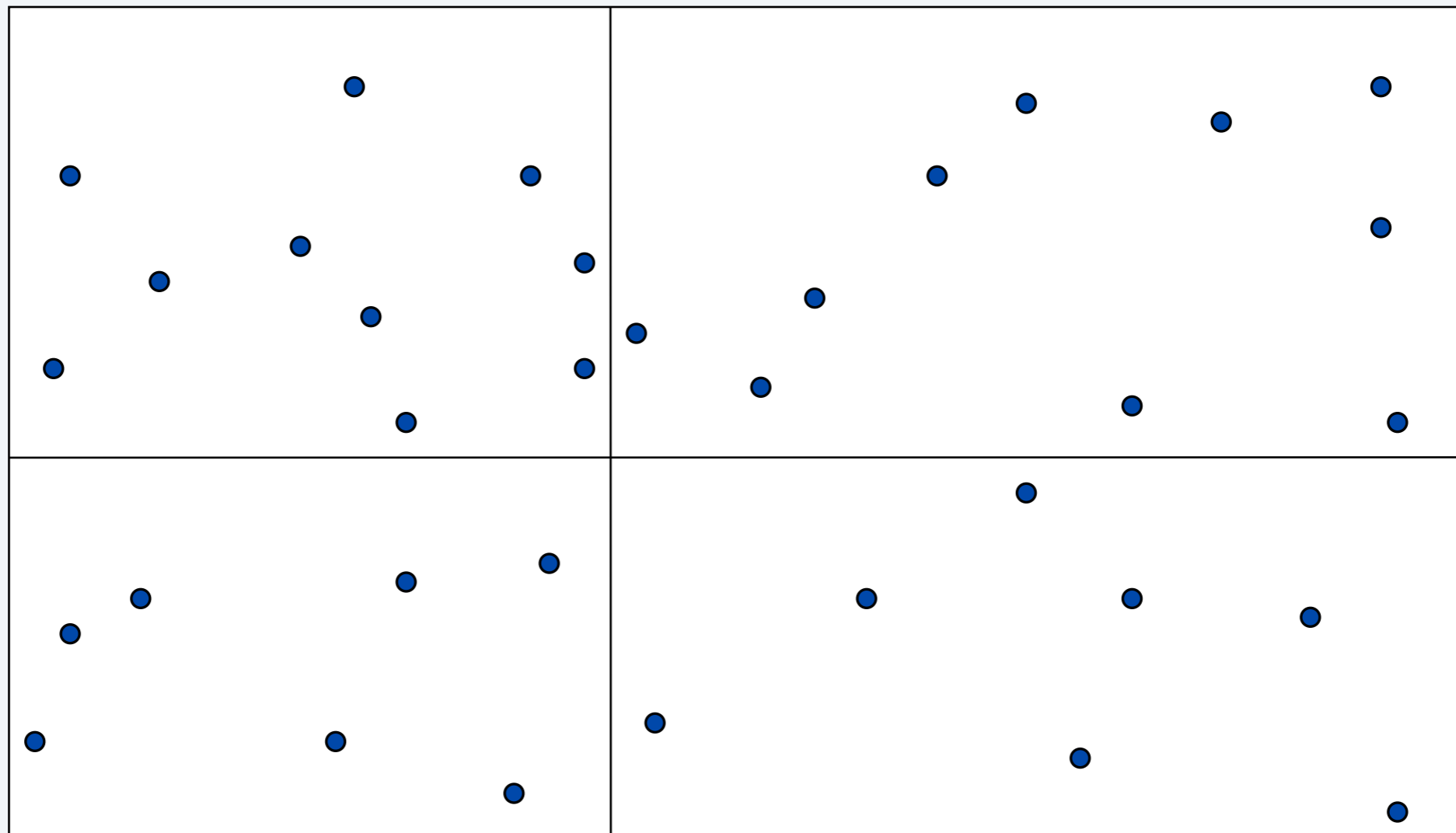
Sorting solution.

- Sort by x -coordinate and consider nearby points.
- Sort by y -coordinate and consider nearby points.



Closest pair of points: second attempt

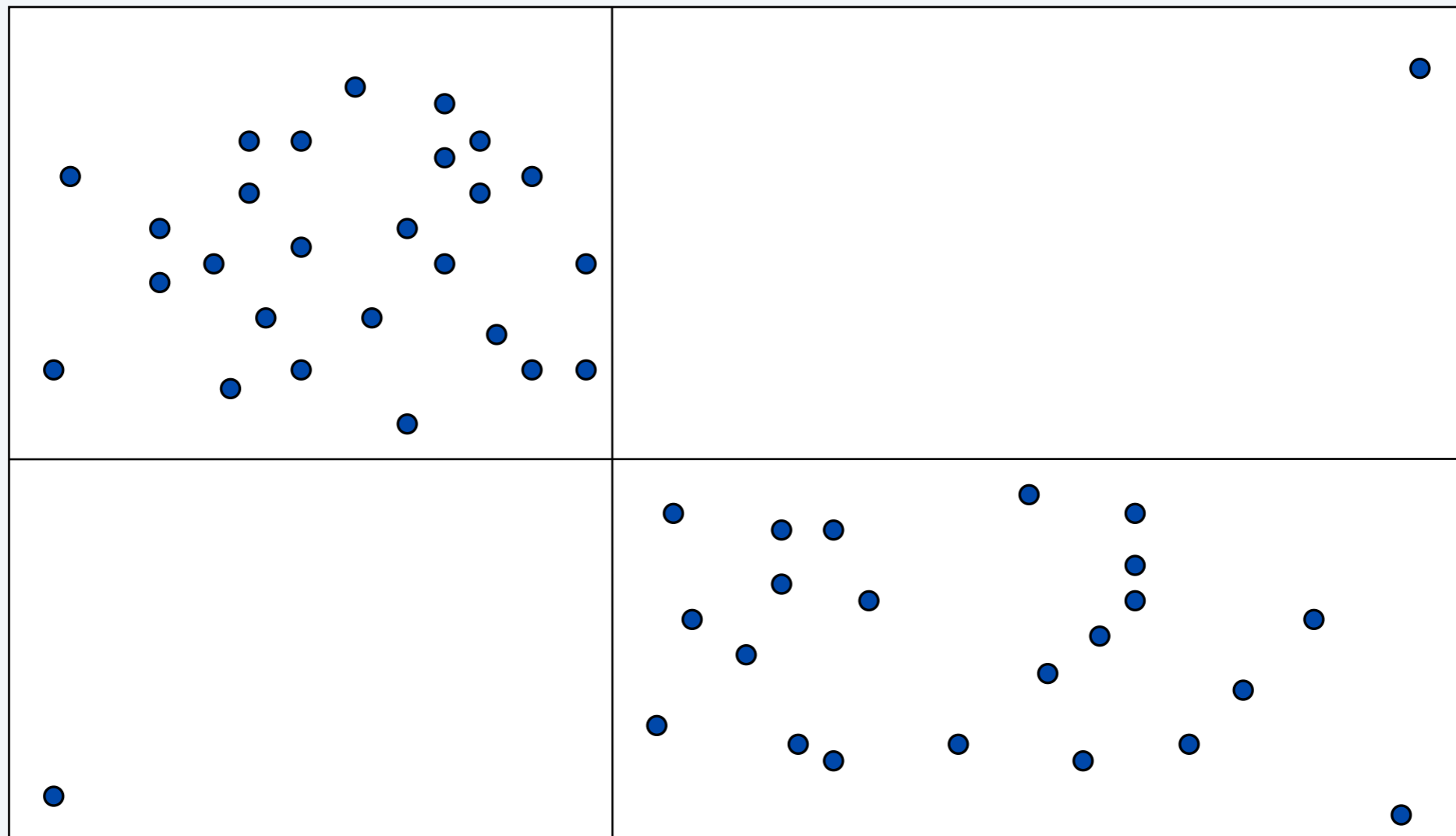
Divide. Subdivide region into 4 quadrants.



Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

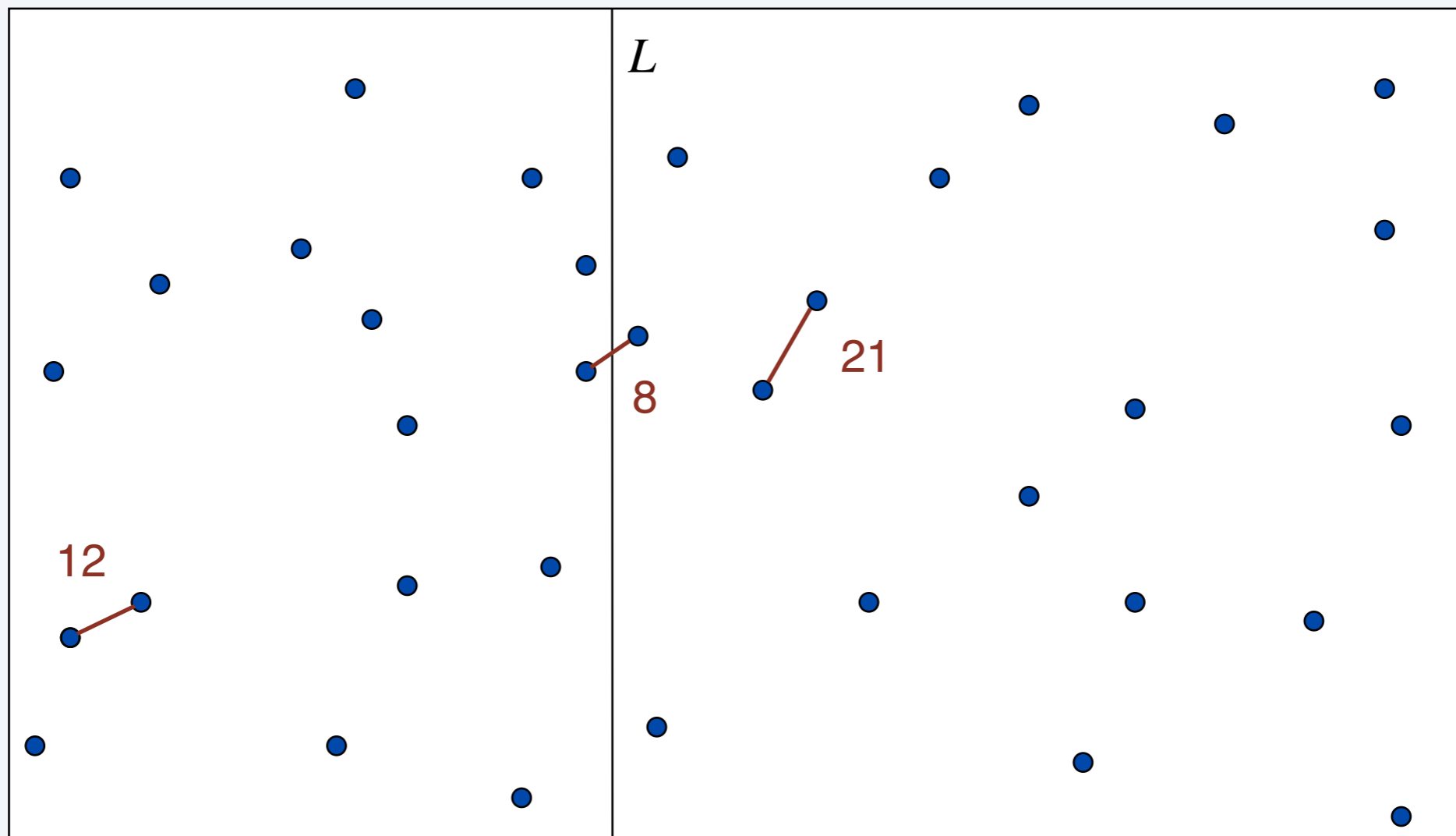
Obstacle. Impossible to ensure $n/4$ points in each piece.



Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- **Combine:** find closest pair with one point in each side.
- Return best of 3 solutions.

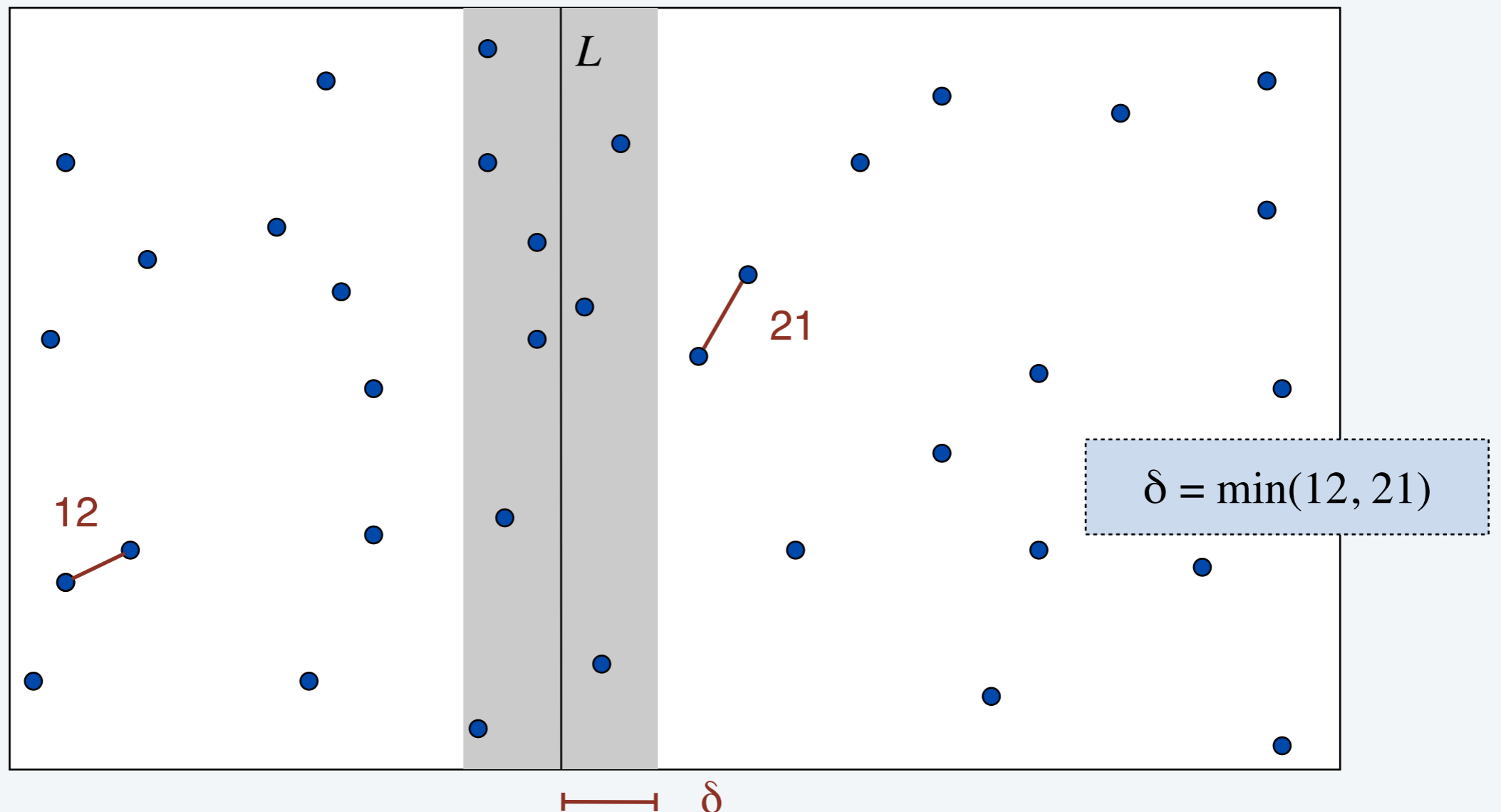
seems like $\Theta(n^2)$



How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance $< \delta$.

- Observation: suffices to consider only those points within δ of line L .

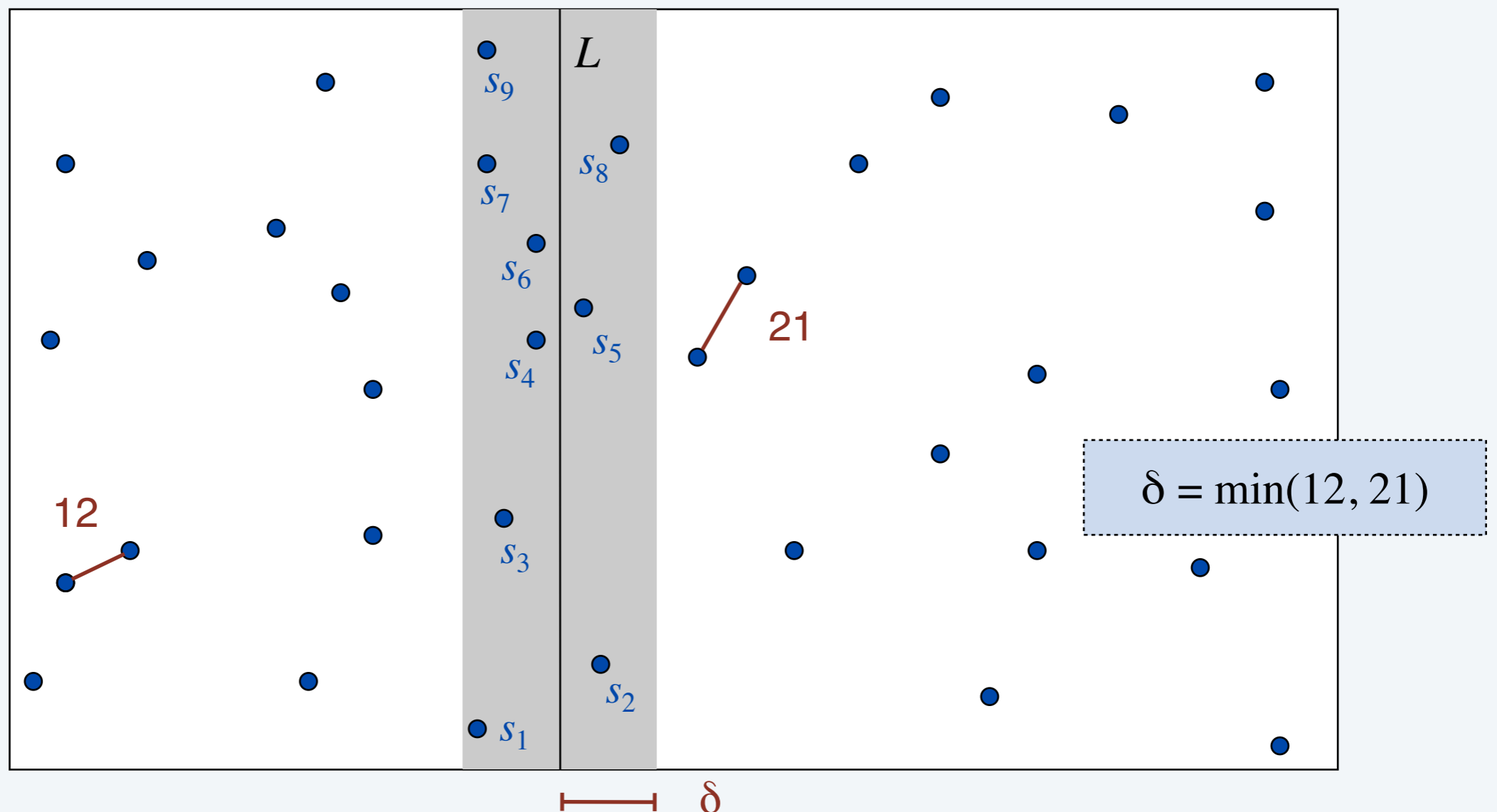


How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance $< \delta$.

- Observation: suffices to consider only those points within δ of line L .
- Sort points in 2δ -strip by their y -coordinate.
- Check distances of only those points within 7 positions in sorted list!

whv?



How to find closest pair with one point in each side?

Def. Let s_i be the point in the 2δ -strip, with the i^{th} smallest y -coordinate.

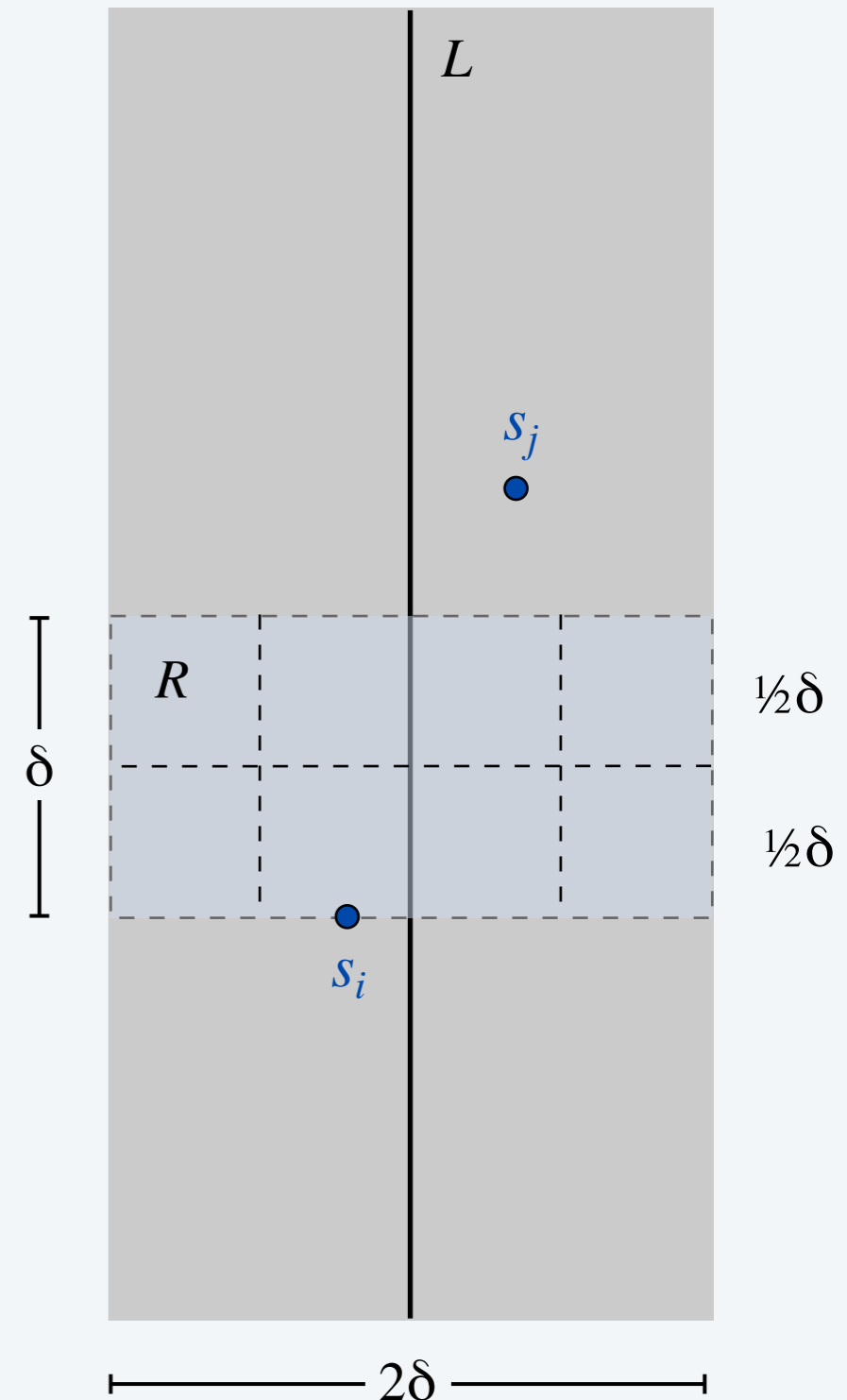
Claim. If $|j - i| > 7$, then the distance between s_i and s_j is at least δ .

Pf.

- Consider the 2δ -by- δ rectangle R in strip whose min y -coordinate is y -coordinate of s_i .
- Distance between s_i and any point s_j above R is $\geq \delta$.
- Subdivide R into 8 squares.
- At most 1 point per square.
- At most 7 points other than s_i can be in R . ■

diameter of square is $\delta / \sqrt{2} < \delta$

constant can be improved with more refined geometric packing argument



Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR(p_1, p_2, \dots, p_n)

Compute vertical line L such that half the points are on each side of the line.

← $O(n)$

$\delta_1 \leftarrow$ **CLOSEST-PAIR**(points in left half).

← $T(\lfloor n / 2 \rfloor)$

$\delta_2 \leftarrow$ **CLOSEST-PAIR**(points in right half).

← $T(\lceil n / 2 \rceil)$

$\delta \leftarrow \min \{ \delta_1, \delta_2 \}$.

$A \leftarrow$ list of all points closer than δ to line L .

← $O(n)$

Sort points in A by y -coordinate.

← $O(n \log n)$

Scan points in A in y -order and compare distance between each point and next 7 neighbors.

← $O(n)$

If any of these distances is less than δ , update δ .

RETURN δ .



What is the solution to the following recurrence?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n \log n) & \text{if } n > 1 \end{cases}$$

- A.** $T(n) = \Theta(n)$.
- B.** $T(n) = \Theta(n \log n)$.
- C.** $T(n) = \Theta(n \log^2 n)$.
- D.** $T(n) = \Theta(n^2)$.

Refined version of closest-pair algorithm

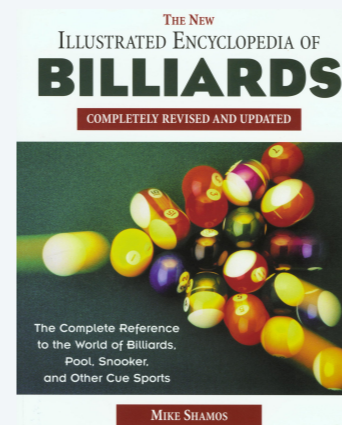
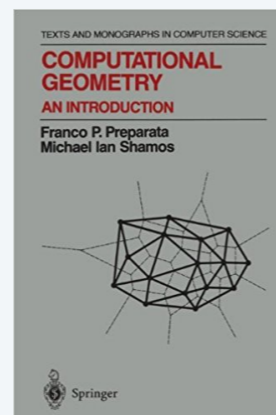
Q. How to improve to $O(n \log n)$?

A. Don't sort points in strip from scratch each time.

- Each recursive call returns two lists: all points sorted by x -coordinate, and all points sorted by y -coordinate.
- Sort by **merging** two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding a closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$





What is the complexity of the 2D closest pair problem?

- A. $\Theta(n)$.
- B. $\Theta(n \log^* n)$.
- C. $\Theta(n \log \log n)$.
- D. $\Theta(n \log n)$.
- E. Not even Tarjan knows.

Computational complexity of closest-pair problem

Theorem. [Ben-Or 1983, Yao 1989] In quadratic decision tree model, any algorithm for closest pair (even in 1D) requires $\Omega(n \log n)$ quadratic tests.

**Lower Bounds for Algebraic Computation Trees
with Integer Inputs***

Andrew Chi-Chih Yao
*Department of Computer Science
Princeton University
Princeton, New Jersey 08544*


$$(x_1 - x_2)^2 + (y_1 - y_2)^2$$


Theorem. [Rabin 1976] There exists an algorithm to find the closest pair of points in the plane whose expected running time is $O(n)$.

A NOTE ON RABIN'S NEAREST-NEIGHBOR ALGORITHM*

Steve FORTUNE and John HOPCROFT
Department of Computer Science, Cornell University, Ithaca, NY, U.S.A.

Received 20 July 1978, revised version received 21 August 1978

Probabilistic algorithms, nearest neighbor, hashing



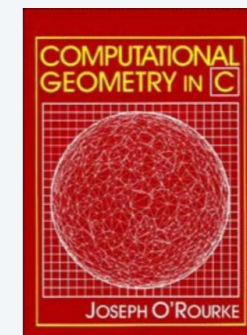
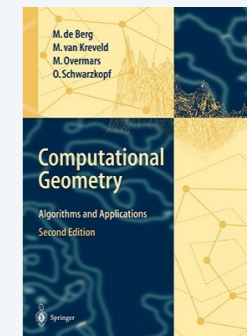
not subject to $\Omega(n \log n)$ lower bound
because it uses the floor function

Digression: computational geometry

Ingenious divide-and-conquer algorithms for core geometric problems.

problem	brute	clever
closest pair	$O(n^2)$	$O(n \log n)$
farthest pair	$O(n^2)$	$O(n \log n)$
convex hull	$O(n^2)$	$O(n \log n)$
Delaunay/Voronoi	$O(n^4)$	$O(n \log n)$
Euclidean MST	$O(n^2)$	$O(n \log n)$

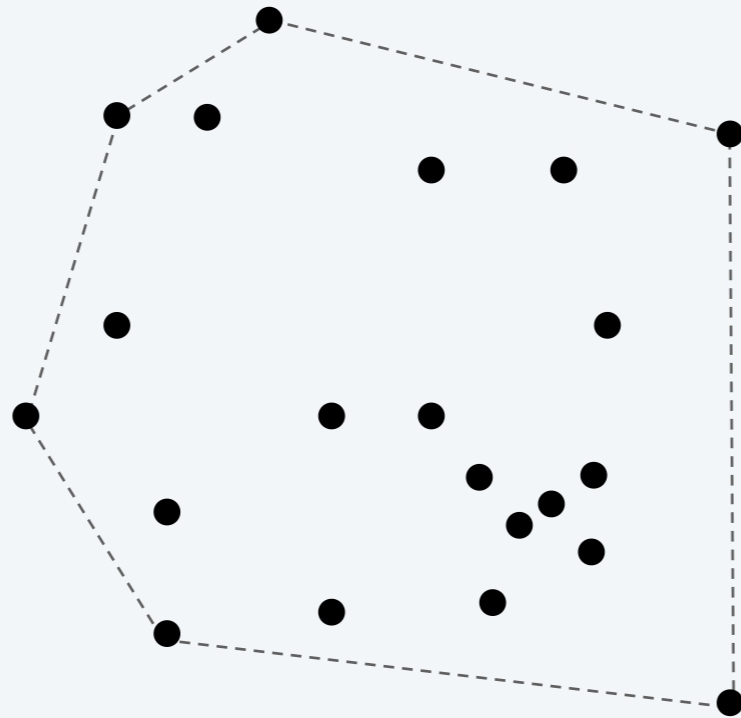
running time to solve a 2D problem with n points



Note. 3D and higher dimensions test limits of our ingenuity.

Convex hull

The **convex hull** of a set of n points is the smallest perimeter fence enclosing the points.

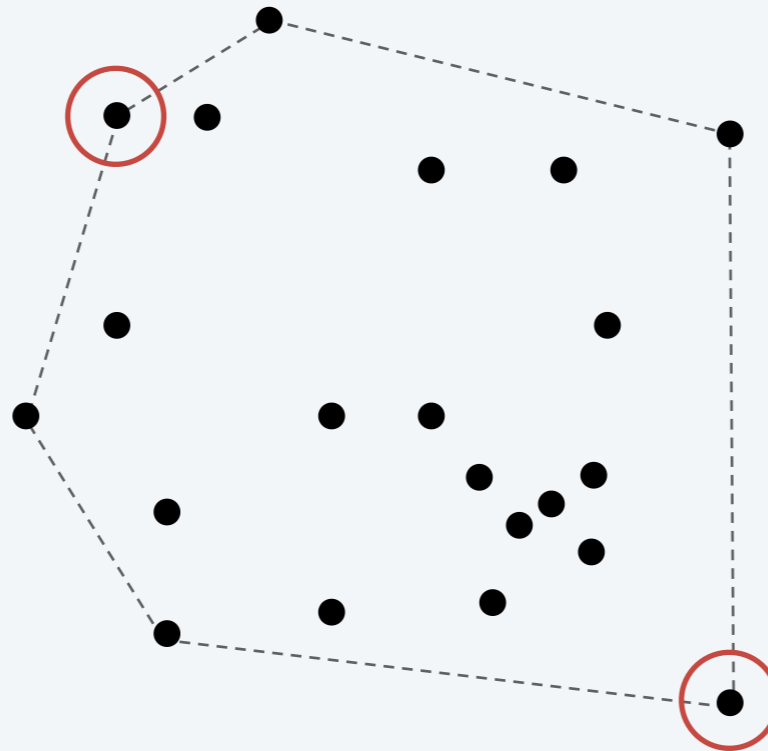


Equivalent definitions.

- Smallest area convex polygon enclosing the points.
- Intersection of all convex set containing all the points.

Farthest pair

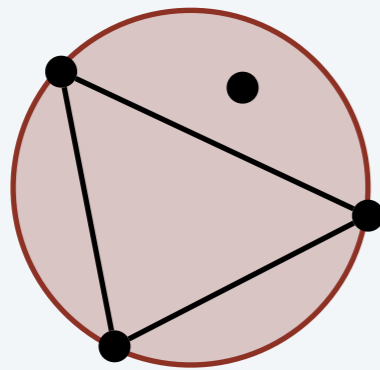
Given n points in the plane, find a pair of points with the largest Euclidean distance between them.



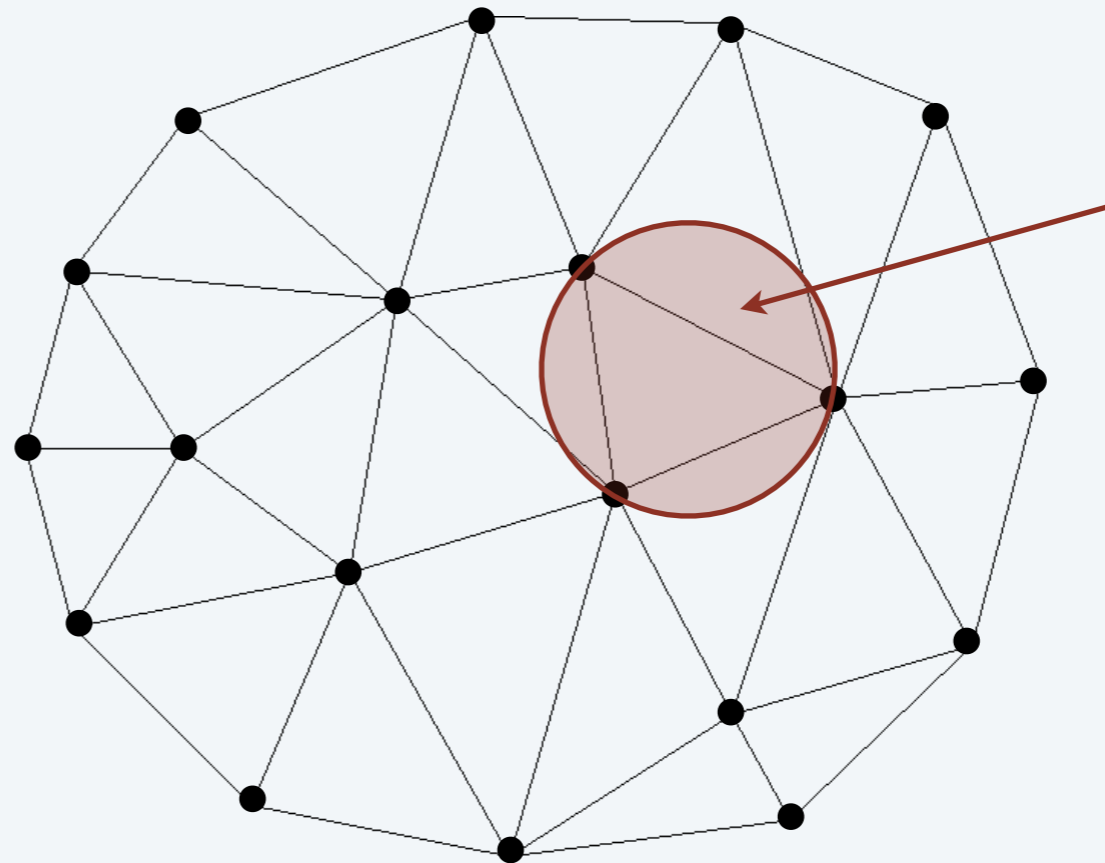
Fact. Points in farthest pair are extreme points on convex hull.

Delaunay triangulation

The **Delaunay triangulation** is a triangulation of n points in the plane such that no point is inside the circumcircle of any triangle.



point inside circumcircle
of 3 points



no point in the set is
inside the circumcircle

Delaunay triangulation of 19 points

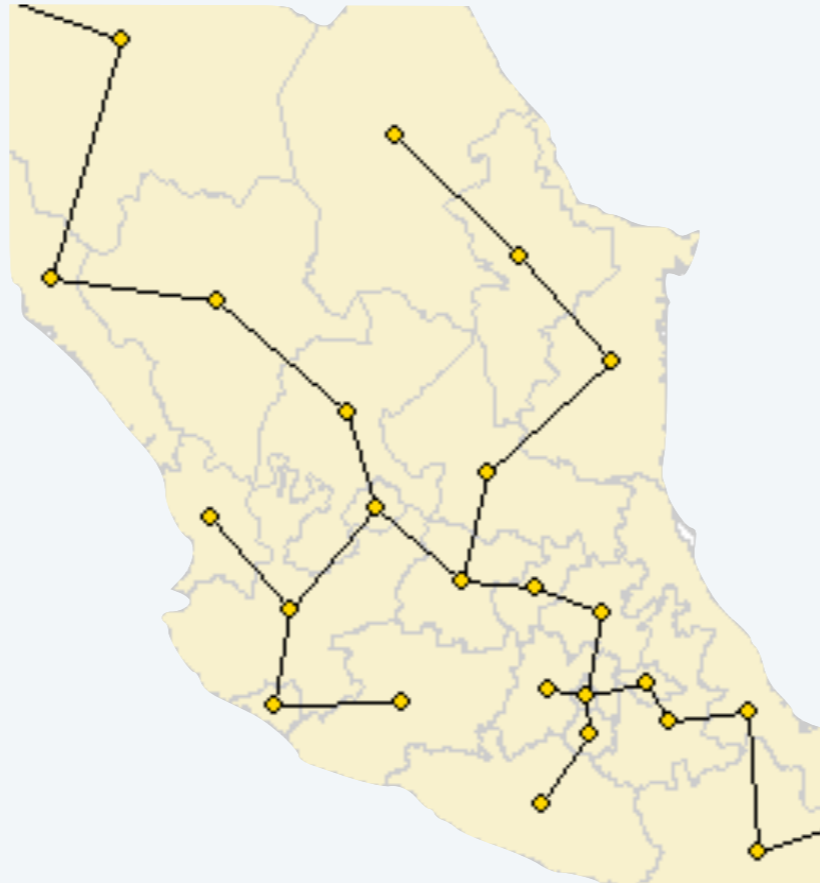
Some useful properties.

- No edges cross.
- Among all triangulations, it maximizes the minimum angle.
- Contains an edge between each point and its nearest neighbor.

Euclidean MST

Given n points in the plane, find MST connecting them.

[distances between point pairs are Euclidean distances]



Fact. Euclidean MST is subgraph of Delaunay triangulation.

Implication. Can compute Euclidean MST in $O(n \log n)$ time.

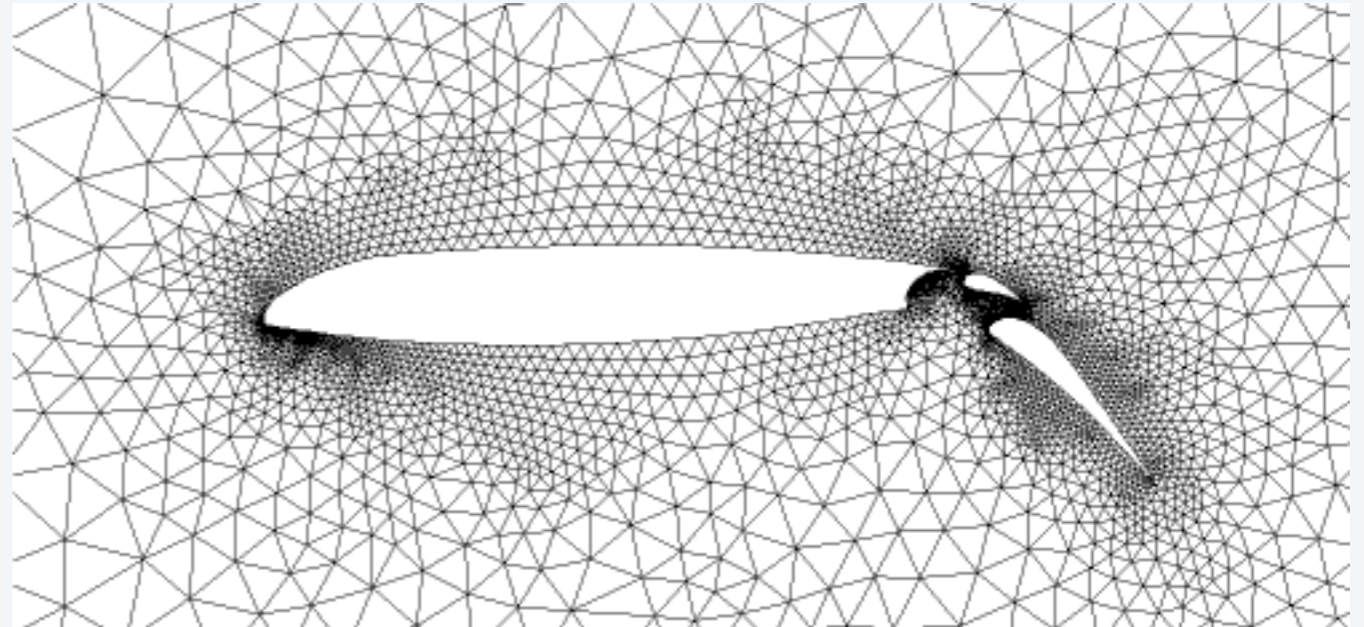
- Compute Delaunay triangulation.
- Compute MST of Delaunay triangulation.

← it's planar
($\leq 3n$ edges)

Computational geometry applications

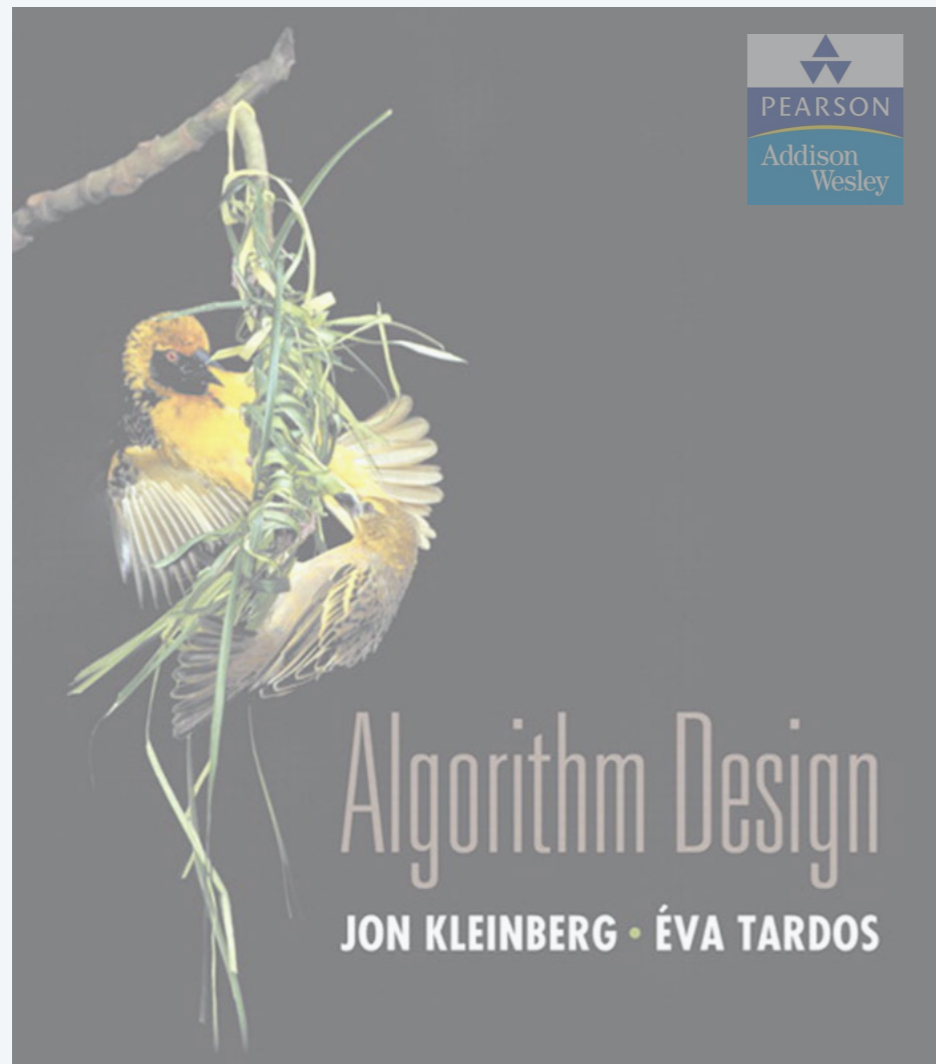
Applications.

- Robotics.
- VLSI design.
- Data mining.
- Medical imaging.
- Computer vision.
- Scientific computing.
- Finite-element meshing.
- Astronomical simulation.
- Models of physical world.
- Geographic information systems.
- Computer graphics (movies, games, virtual reality).



airflow around an aircraft wing

<http://www.ics.uci.edu/~eppstein/geom.html>



SECTION 5.5

5. DIVIDE AND CONQUER

- ▶ *mergesort*
- ▶ *Master Theorem*
- ▶ *randomized quicksort*
- ▶ *median and selection*
- ▶ *counting inversions*
- ▶ *closest pair of points*
- ▶ ***integer multiplication***

Integer addition and subtraction

Addition. Given two n -bit integers a and b , compute $a + b$.

Subtraction. Given two n -bit integers a and b , compute $a - b$.

Grade-school algorithm. $\Theta(n)$ bit operations.

← “bit complexity”
(instead of word RAM)

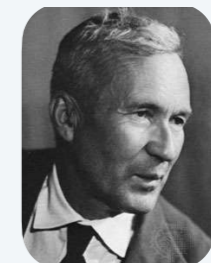
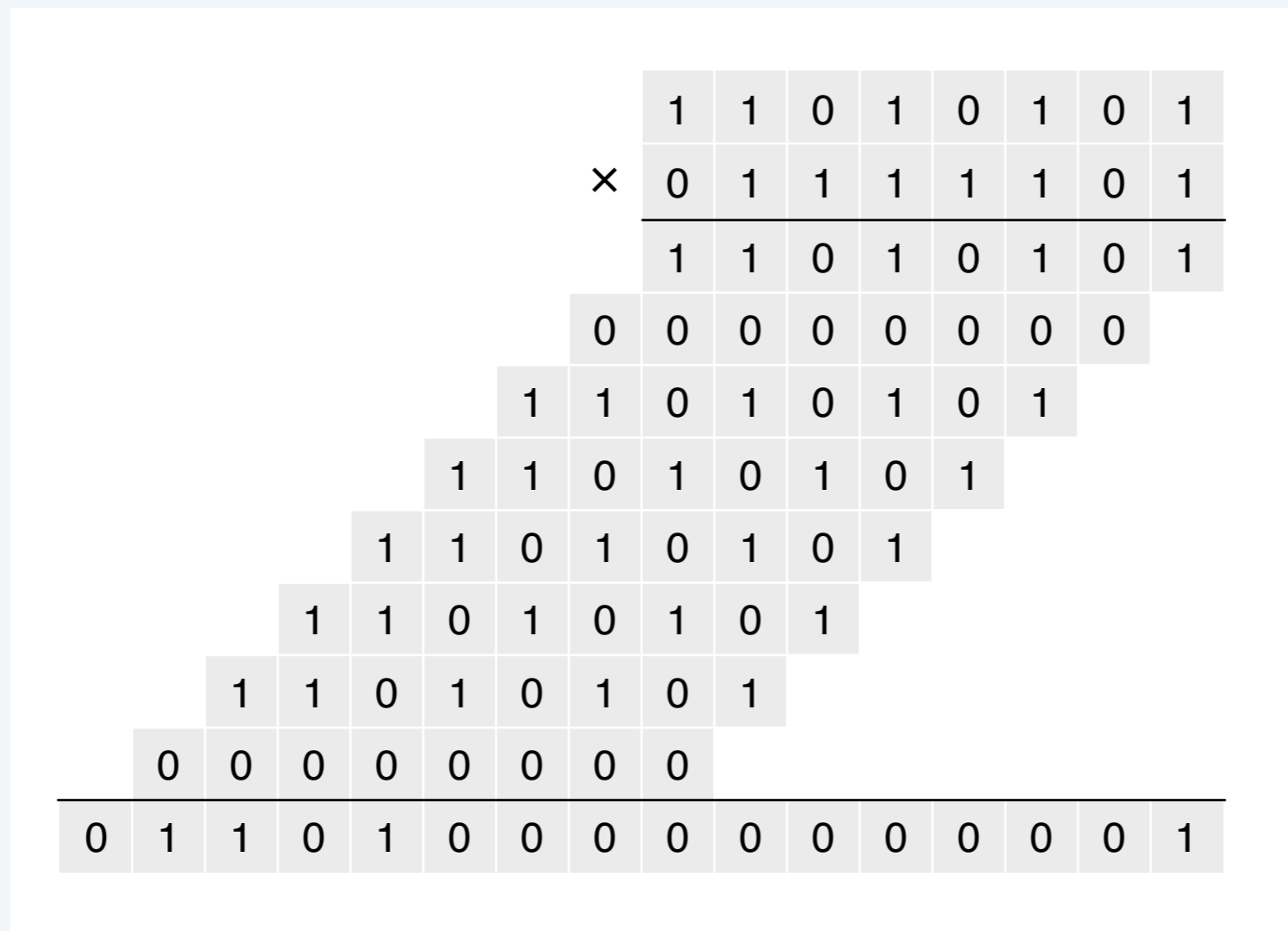
	1	1	1	1	1	1	0	1	
		1	1	0	1	0	1	0	1
+		0	1	1	1	1	1	0	1
	1	0	1	0	1	0	0	1	0

Remark. Grade-school addition and subtraction algorithms are optimal.

Integer multiplication

Multiplication. Given two n -bit integers a and b , compute $a \times b$.

Grade-school algorithm (long multiplication). $\Theta(n^2)$ bit operations.



Conjecture. [Kolmogorov 1956] Grade-school algorithm is optimal.

Theorem. [Karatsuba 1960] Conjecture is false.

Divide-and-conquer multiplication

To multiply two n -bit integers x and y :

- Divide x and y into low- and high-order bits.
- Multiply **four** $\frac{1}{2}n$ -bit integers, recursively.
- Add and shift to obtain result.

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

← use bit shifting
to compute 4 terms

$$x y = (2^m a + b) (2^m c + d) = 2^{2m} \underbrace{ac}_1 + 2^m (\underbrace{bc}_2 + \underbrace{ad}_3) + \underbrace{bd}_4$$

Ex. $x = \underbrace{1000}_a \underbrace{1101}_b \quad y = \underbrace{11100000}_c \underbrace{001}_d$

Divide-and-conquer multiplication

MULTIPLY(x, y, n)

IF ($n = 1$)

RETURN $x \times y$.

ELSE

$m \leftarrow \lceil n / 2 \rceil$.

$a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$.

$c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

$e \leftarrow \text{MULTIPLY}(a, c, m)$.

$f \leftarrow \text{MULTIPLY}(b, d, m)$.

$g \leftarrow \text{MULTIPLY}(b, c, m)$.

$h \leftarrow \text{MULTIPLY}(a, d, m)$.

RETURN $2^{2m} e + 2^m (g + h) + f$.

← $\Theta(n)$

← $4 T(\lceil n / 2 \rceil)$

← $\Theta(n)$



How many bit operations to multiply two n -bit integers using the divide-and-conquer multiplication algorithm?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 4T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- A.** $T(n) = \Theta(n^{1/2})$.
- B.** $T(n) = \Theta(n \log n)$.
- C.** $T(n) = \Theta(n^{\log_2 3}) = O(n^{1.585})$.
- D.** $T(n) = \Theta(n^2)$.

Karatsuba trick

To multiply two n -bit integers x and y :

- Divide x and y into low- and high-order bits.
- To compute middle term $bc + ad$, use identity:

$$bc + ad = ac + bd - (a - b)(c - d)$$

- Multiply only **three** $\frac{1}{2}n$ -bit integers, recursively.

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

middle term
↓

$$x = \underbrace{1000}_a \underbrace{1101}_b$$

$$y = \underbrace{1110000}_c \underbrace{001}_d$$

$$x y = (2^m a + b) (2^m c + d) = 2^{2m} ac + 2^m (bc + ad) + bd$$

$$= 2^{2m} ac + 2^m (ac + bd - (a - b)(c - d)) + bd$$

1

1

3

2

3

Karatsuba multiplication

KARATSUBA-MULTIPLY(x, y, n)

IF ($n = 1$)

RETURN $x \times y$.

ELSE

$m \leftarrow \lceil n / 2 \rceil$.

$a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$. ← $\Theta(n)$

$c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

$e \leftarrow \text{KARATSUBA-MULTIPLY}(a, c, m)$.

$f \leftarrow \text{KARATSUBA-MULTIPLY}(b, d, m)$. ← $3 T(\lceil n / 2 \rceil)$

$g \leftarrow \text{KARATSUBA-MULTIPLY}(|a - b|, |c - d|, m)$.

 Flip sign of g if needed.

RETURN $2^{2m} e + 2^m (e + f - g) + f$. ← $\Theta(n)$

Karatsuba analysis

Proposition. Karatsuba's algorithm requires $O(n^{1.585})$ bit operations to multiply two n -bit integers.

Pf. Apply Case 3 of the master theorem to the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 3T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$

$$\implies T(n) = \Theta(n^{\log_2 3}) = O(n^{1.585})$$


Practice.

- Use base 32 or 64 (instead of base 2).
- Faster than grade-school algorithm for about 320–640 bits.

Integer arithmetic reductions

Integer multiplication. Given two n -bit integers, compute their product.

arithmetic problem	formula	bit complexity
integer multiplication	$a \times b$	$M(n)$
integer square	a^2	$\Theta(M(n))$
integer division	$\lfloor a / b \rfloor, a \bmod b$	$\Theta(M(n))$
integer square root	$\lfloor \sqrt{a} \rfloor$	$\Theta(M(n))$

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$


integer arithmetic problems with the same bit complexity $M(n)$ as integer multiplication

History of asymptotic complexity of integer multiplication

year	algorithm	bit operations
12xx	grade school	$O(n^2)$
1962	Karatsuba-Ofman	$O(n^{1.585})$
1963	Toom-3, Toom-4	$O(n^{1.465}), O(n^{1.404})$
1966	Toom-Cook	$O(n^{1+\epsilon})$
1971	Schönhage-Strassen	$O(n \log n \cdot \log \log n)$
2007	Fürer	$n \log n 2^{O(\log^* n)}$
2019	Harvey-van der Hoeven	$O(n \log n)$
	???	$O(n)$

number of bit operations to multiply two n -bit integers

Remark. GNU Multiple Precision library uses one of first five algorithms depending on n .



used in Maple, Mathematica, gcc, cryptography, ...

GMP
«Arithmetic without limitations»